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Peg solitaire on graphs with jumping and merging allowed. (English. English summary)

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The following describes *peg solitaire*. Consider a connected, undirected graph $G = (V, E)$. Put pegs in all of the vertices except one. This is the *starting state*. From this state, a process is followed. At each step in the process, consider some set of three vertices u, v, w such that u neighbors v and v neighbors w. There are two possible types of move.

A *jump*: if u and v have pegs and w has no peg, we can move the peg from u to w, removing and discarding the peg at v, which has been "jumped over".

A *merge*: if u and w have pegs and v has no peg, we discard the pegs at u and w, and add a peg at v, created by the other pegs "merging".

A *terminal state* is a state such that no more jumps or merges are possible.

G is *solvable* if there is some starting state such that a terminal state with only a single peg can be reached.

G is *freely solvable* if, from every starting state, a terminal state with only a single peg can be reached.

The paper under review shows that several classes of graph are solvable, including stars, caterpillars (obtained by adding leaves to a path) and trees of diameter 4 and 5.

Theorem 2.1 shows that star graphs are freely solvable. This is interesting because if we allow only jumps or only merges, the star is unsolvable. In contrast, when both are permitted, as long as there are at least two remaining pegs, there is either a hole at the central vertex, in which case a merge move is available, or a peg at the central vertex, in which case a jump move is available.

A caterpillar is constructed by starting with a path on vertices $(x_1, \dots, x_n)$ and, for $i = 1, \dots, n$, adding leaves $x_{i,1}, \dots, x_{i,a_i}$, $a_i \geq 1$, connected to x_i. Theorem 3.1 shows that such caterpillars are solvable. Theorem 3.2 extends the result for some cases in which $a_i = 0$ for some vertices on the initial path.

The proofs use packages and purges. A *package* is a subgraph that has a specific configuration of pegs and holes. A *purge* is a sequence of moves which preserves the locations of certain pegs and holes while removing the remaining pegs on the subgraph. Purges using the star subgraph are used in the proofs of Theorems 3.1 and 3.2.

Theorem 4.1 shows that trees of diameter 4 are solvable. It is proved using star purges as well as two more purges on trees (trident purges and wishbone purges). Theorem 4.2 combines the results on stars, caterpillars and trees of diameter 4 to show that trees of diameter 5 are solvable. *Jonathan Newton*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.