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Hat guessing on books and windmills. **(English summary)**

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The *hat guessing game* involves n players who are identified with the vertices of a graph. Each player wears a hat which is one of q possible colors. Players can observe the colors of the hats worn by their neighbors on the graph but cannot observe the color of their own hat. All of the players simultaneously guess their own hat color according to a predetermined strategy which can depend on the colors of the hats worn by their neighbors. The *hat guessing number* $HG(G)$ of a graph G is the largest integer q such that there exists a guessing strategy which guarantees that at least one player guesses correctly no matter which colors are assigned.

A *book* $B_{d,n}$ is a graph with two sets of vertices, the *spine* V_d and *pages* V_n, of size d and n respectively, such that the induced graph on V_d is a complete subgraph, and each vertex in V_n is adjacent to every vertex in V_d but to no other vertices.

Theorem 2 shows that, for sufficiently large n, we have $HG(B_{d,n}) = 1 + \sum_{i=1}^{d} i^i$. The proof proceeds as follows. A set $S \subset \mathbb{N}^d$ is *coverable* if there exists a partition $S_1 \sqcup \cdots \sqcup S_d$ such that S_i contains at most one point along any line parallel to the i-th coordinate axis. It was shown in [X. He and R. Li, Electron. J. Combin. **27** (2020), no. 3, Paper No. 3.58; MR4245171] that, for large enough n, $HG(B_{d,n})$ is the size of the smallest non-coverable set in d dimensions. In the current paper, coverability is reformulated as a matching condition and P. Hall's Marriage Theorem [J. London Math. Soc. **10** (1935), 26–30, doi:10.1112/jlms/s1-10.37.26] is used to show that $S \subset \mathbb{N}^d$ is coverable if and only if S is *numerically coverable* (Lemma 8), where S is numerically coverable if $\sum_{i=1}^{d} |\pi_i(S)| \geq |S|$, where $\pi_i(S)$ is the $(d-1)$-dimensional projection of S onto the i-th coordinate hyperplane.

The next step shows that all small enough sets are numerically coverable, thus proving a lower bound on $HG(B_{d,n})$. Combined with an upper bound from [M. Gadouleau, SIAM J. Discrete Math. **32** (2018), no. 3, 1922–1945; MR3835237], this proves the theorem.

Theorem 3 shows that for the complete bipartite graph $K_{3,3}$, we have $HG(K_{3,3}) = 3$. A key part of the proof (Lemma 13) shows that $HG(K_{3,3}) \geq 4$ if and only if there exist three partitions P, Q, R of $[4]^3$,

$$[4]^3 = P_1 \sqcup P_2 \sqcup P_3 \sqcup P_4 = Q_1 \sqcup Q_2 \sqcup Q_3 \sqcup Q_4 = R_1 \sqcup R_2 \sqcup R_3 \sqcup R_4,$$

such that $P_i \cup Q_j \cup R_k$ contains a $3 \times 3 \times 3$ cube for all choices of $1 \leq i, j, k \leq 4$.

The proof then proceeds to show that such partitions cannot exist, so therefore $HG(K_{3,3}) \leq 3$. The proof is completed by noting that as $K_{2,2}$ is a subgraph of $K_{3,3}$ and $HG(K_{2,2}) = 3$, it must be that $HG(K_{3,3}) \geq 3$ as well.

In addition, similar techniques are used to determine $HG(\cdot)$ for most *windmill* graphs, graphs composed of complete subgraphs that share a single vertex. *Jonathan Newton*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.