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Hat guessing numbers of strongly degenerate graphs. (English. English summary)

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The *hat guessing game* involves n players who are identified with the vertices of a graph. Each player wears a hat which is one of q possible colors. Players can observe the colors of the hats worn by their neighbors on the graph but cannot observe the color of their own hat. All of the players simultaneously guess their own hat color according to a predetermined strategy which can depend on the colors of the hats worn by their neighbors. The *hat guessing number* $HG(G)$ of a graph G is the largest integer q such that there exists a guessing strategy which guarantees that at least one player guesses correctly no matter which colors are assigned.

For a strictly positive integer d, a vertex v is *d-removable* in G if (i) v has degree at most d in G and (ii) v has at most one neighbor in G with degree strictly more than d. G is *strongly d-degenerate* if every nonempty subgraph G' of G contains a vertex which is d-removable in G'.

The main result of the paper [Theorem 1.2] is that every strongly d-degenerate graph G satisfies $HG(G) \leq (2d)^d$.

The proof involves considering an extension of the hat guessing game in which a player may guess more than one color for his hat. For a given graph, allowing multiple guesses cannot decrease the hat guessing number, so any upper bound on $HG(G)$ for a game with multiple guesses is also a bound on the game with a single guess. Thus, the proof proceeds by showing a bound on specific games with multiple guesses.

Consider G such that vertices with degree more than d have a single guess and vertices with degree $d_G(v) \leq d$ have $(2d)^{d-d_G(v)}$ guesses.

The proof is by induction on the number of vertices in G.

For a single vertex/player, we have $d_G(v) = 0$, so this player has $(2d)^d$ guesses, implying a hat guessing number of $(2d)^d$.

The induction step involves removing a d-removable vertex w from G to obtain G' and updating the number of guesses as above. This gives additional guesses to neighbors of w in G; as for v neighboring w in G, we have $d_{G'}(v) < d_G(v)$ and therefore

$$(2d)^{d-d_{G'}(v)} > (2d)^{d-d_G(v)}.$$

It is shown that these additional guesses are sufficient and that moving from the game with G to the game with G' does not decrease the hat guessing number. Therefore an upper bound on the hat guessing number that applies to G' also applies to G. This completes the induction. *Jonathan Newton*

[References]

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.