MR4585212 91A12

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Core stability of the Shapley value for cooperative games. (English. English summary)

Soc. Choice Welf. **60** (2023), *no.* 4, 523–543.

This paper provides some characterizations for cooperative games in terms of polyhedral cones. Consider a cooperative game given by a set of players $N=\{1,\dots,n\}$ and a characteristic function $v\colon 2^N\to\mathbb{R}$, $v(\varnothing)=0$. Define

$$u_S^-(T)=\begin{cases} -1, & \text{if } T=S,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$u_S(T)=\begin{cases} 1, & \text{if } S\subseteq T,\\ 0, & \text{otherwise.}\end{cases}$$

Let σ be a permutation of N and, for $i\in N$, let ρ_i^σ be the set of predecessors of i in σ. Let $mc_{i,\sigma}(v)=v(\rho_i^\sigma\cup\{i\})-v(\rho_i^\sigma)$. Let Π be the set of all permutations. The Shapley value $Sh(v)$ of v is given by

$$Sh_i(v)=\frac{1}{n!}\sum_{\sigma\in\Pi} mc_{i,\sigma}(v)$$

and the Core $C(v)$ by

$$C(v)=\left\{x\in\mathbb{R}^n:\sum_{j\in N}x_j=v(N)\text{ and }\sum_{j\in S}x_j\geq v(S)\text{ for all }S\subseteq N\right\}.$$

Theorem 2 of the paper characterizes the set of games with a nonempty core as those expressible as a linear combination

$$v=\sum_{i\in N}\alpha_i u_{\{i\}}+\sum_{\varnothing\neq S\subsetneq N}\alpha_S^- u_S^-$$

for $(\alpha_i)_{i\in N}\in\mathbb{R}^n$ and $(\alpha_S^-)_{S\subsetneq N}\geq\mathbf{0}$. Furthermore, if $x\in C(v)$, then there exists such a combination with $\alpha_i=x_i$ for all $i\in N$.

It follows that if $Sh(v)\in C(v)$, then the game can be expressed as

$$v=\sum_{i\in N}Sh_i(v)u_{\{i\}}+\sum_{\varnothing\neq S\subsetneq N}\alpha_S^- u_S^-.$$

Furthermore, by construction,

$$Sh\left(\sum_{\varnothing\neq S\subsetneq N}\alpha_S^- u_S^-\right)=0,$$

and as a consequence the second term can be written as a linear combination of a specific type of basic vector. This characterization of games such that $Sh(v)\in C(v)$ is Theorem 4 of the paper. *Jonathan Newton*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.