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Cautious farsighted stability in network formation games with streams of payoffs. (English summary)
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Consider a set of vertices that represent players. Let $\mathbb{G}$ be the set of simple, undirected graphs on these vertices. A *path* from graph g to g' is a sequence of graphs in $\mathbb{G}$, beginning with g and ending at g', such that each graph in the sequence differs from its predecessor by at most one edge. Let $\mathbb{P}$ be the set of all possible paths for all possible g and g'. For a path $P=(g_1,\dots,g_K)$, define the continuation from k as $P_k=(g_k,\dots,g_K)$. Let g^m denote a path that consists of m repetitions of g.

Each player/vertex i has a payoff $\pi_i(P)\in\mathbb{R}$ for every path $P\in\mathbb{P}$.

Path $P=(g_1,\dots,g_K)$ is an *improving path* if, for all $1\le k\le K-1$, either (a) or (b) holds.

(a) There is an edge between some i and j that is present in g_k but not in g_{k+1}. $\pi_i(P_{k+1})>\pi_i(g_k^{|P_{k+1}|})$ or $\pi_j(P_{k+1})>\pi_j(g_k^{|P_{k+1}|})$.

(b) There is an edge between some i and j that is present in g_{k+1} but not in g_k. $\pi_i(P_{k+1})>\pi_i(g_k^{|P_{k+1}|})$ and $\pi_j(P_{k+1})\ge\pi_j(g_k^{|P_{k+1}|})$.

Let $P^I(g)$ denote the set of all improving paths starting at network g.

Path $P=(g_1,\dots,g_K)$ is a *surely improving path* relative to $G\subseteq\mathbb{G}$ if, for all $1\le k\le K-1$, either (c) or (d) holds.

(c) There is an edge between some i and j that is present in g_k but not in g_{k+1}. $\pi_i(\tilde{P})>\pi_i(g_k^{|\tilde{P}|})$ for any $\tilde{P}\in P^I(g_{k+1})$ ending at a graph in G, or $\pi_j(\tilde{P})>\pi_j(g_k^{|\tilde{P}|})$ for any $\tilde{P}\in P^I(g_{k+1})$ ending at a graph in G.

(d) There is an edge between some i and j that is present in g_{k+1} but not in g_k. $\pi_i(\tilde{P})\ge\pi_i(g_k^{|\tilde{P}|})$ and $\pi_j(\tilde{P})\ge\pi_j(g_k^{|\tilde{P}|})$, with at least one inequality strict, for any $\tilde{P}\in P^I(g_{k+1})$ ending at a graph in G.

Let $P^{SI}(g,G)$ denote the set of all surely improving paths relative to G that start at network g. By definition $P^{SI}(g,G)\subseteq P^I(g)$ for any $G\subseteq\mathbb{G}$.

The definition of a surely improving path is restrictive in that it requires that any player who moves the graph from g_k to g_{k+1} must prefer *any* improving path starting from g_{k+1} to remaining at g_k. That is, they do not need to be confident that subsequent changes in the graph will follow any specific improving path.

A set $G\subset\mathbb{G}$ is *cautious path stable* if (1) for all $g'\in\mathbb{G}\setminus G$, there exists $P\in P^{SI}(g',G)$ such that P ends at a graph in G, and (2) for all $G'\subsetneq G$, G' does not satisfy (1). That is, a cautious path stable set G can always be reached from outside of G via a surely improving path, but this is not true for any strict subset of G. The set G is interpreted as a 'cautious' prediction insofar as it is defined using a relation, the existence of a surely improving path, that is quite restrictive.

Proposition 1 shows that a cautious path stable set always exists. The proof proceeds by noting that $\mathbb{G}$ trivially satisfies (1). Any set G' that satisfies (1) either satisfies (2) or not. If not, then take $G''\subsetneq G'$ that satisfies (1). Proceed until obtaining a G that satisfies (2). Note that this must occur, as the sets are strictly decreasing in size and any set with a single element trivially satisfies (2).

The remainder of the paper considers some properties of cautious path stable sets, gives conditions for uniqueness and solves some examples. *Jonathan Newton*