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Convergence in games with continua of equilibria. (English summary)
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Consider an undirected, unweighted graph. Each vertex of the graph is associated with a *player*. Each player can produce a non-negative real quantity of a public good. Take a given player i. If the sum of the quantities produced by the neighbors (on the graph) of player i is at least 1, then the *best response* of player i is to produce a quantity 0. If the sum of the quantities produced by the neighbors of player i is $X_{-i} < 1$, then the best response of player i is to produce a quantity $x_i = 1 - X_{-i}$. When every player produces a quantity that is a best response, then we have a *Nash equilibrium*. Note that there will typically be a continuum of Nash equilibria in such a setting.

Writing $x(t)$ as the vector of quantities produced at time t and $\mathrm{Br}(x(t))$ as the vector of best responses to $x(t)$, we can write the continuous-time best response dynamics $\dot{x}(t) = \mathrm{Br}(x(t)) - x(t)$. The paper gives a method for showing convergence of $x(t)$ to a Nash equilibrium as $t \to \infty$. The authors note that there are many methods to prove convergence when equilibria are isolated but not when they are part of a continuum, in which case methods are scarce and generally difficult to apply.

Note that convergence to Nash equilibrium is contrasted with convergence to the set of Nash equilibria, which is typically easy to show. However, it is possible that $x(t)$ might converge to the set of Nash equilibria without ever converging to any specific member of that set.

To illustrate the method, the paper considers a graph of two players, players 1 and 2, with an edge between them. The best response of player 1 is $1 - x_2$. The best response of player 2 is $1 - x_1$. The set of Nash equilibria is thus all x such that $x_1 + x_2 = 1$. Furthermore, the continuous-time best response dynamic gives $\dot{x}_1 = \dot{x}_2 = (1 - x_1 - x_2)$.

The behavior of the process close to a given Nash equilibrium is then considered. Specifically, a comparison is made between the directions in which the dynamic can move when it is close to x (the *direction cone*) and the directions in which the dynamic could move while remaining close to the set of Nash equilibria (the *tangent cone*).

In this two-player case, by considering $\dot{x}_1 = \dot{x}_2 = (1 - x_1 - x_2)$, we see that the direction cone is the set of vectors proportional to $(1, 1)$. In contrast, by considering the set of Nash equilibria, we see that the tangent cone is the set of vectors proportional to $(1, -1)$. Thus, the dynamic cannot move away from x while remaining close to the set of Nash equilibria. Therefore, convergence to the set of Nash equilibria implies convergence to a Nash equilibrium.

The paper uses this method to show convergence for the general case of the problem (> 2 players, arbitrary graph). This involves further technical difficulties, and finding a convenient expression for the tangent cone is not always straightforward.

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