International Economic Review

WILEY

ORIGINAL ARTICLE OPEN ACCESS

Pairwise Imitation and Tournament Graphs

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Received: 19 January 2026 | **Accepted:** 19 January 2026

Keywords: contests | imitation | one-shot | spite | stability

ABSTRACT

This paper investigates strategic dynamics under the behavioral rule of pairwise interact and imitate (PII), which requires minimal information and emphasizes outperforming opponents in pairwise interactions. We characterize PII using weak tournament graphs and, for a broad class of dynamics, establish a one-shot stability result for stochastic stability. Applications include Cournot competition, strategic complements and substitutes, externalities, Nash demand games, and status-seeking contests. The analysis highlights the competing roles of spite effects and perturbations in favoring relative success versus efficiency.

JEL Classification: C71, C73, D01

1 | Introduction

Alice is occasionally matched to opponents in some strategic setting. She would like to obtain high payoffs in such interactions, but only occasionally updates her strategy and, when she does, she does not think too hard about it. Specifically, following an interaction, she may sometimes imitate the opponent's strategy, but only if she obtained a lower payoff than the opponent in the interaction. If she obtained a higher payoff, she will never imitate. This behavioral rule requires very little information. Alice requires no memory of past interactions. Her changes in strategy depend only on the current strategies and realized payoffs of herself and her opponent. No hypothetical reasoning about other strategies and payoffs is involved. As such, this behavioral rule is appropriate for situations in which agents have limited cognitive ability or have a particular interest in "beating" opponents in pairwise interactions. Such situations arise when relative status is important, such as in the competition for promotions in the workplace, displays of conspicuous consumption, competition to be chosen as a social partner, and industries where rapid growth relative to others is decisive. This boundedly rational updating rule is known as pairwise interact and imitate (PII, Bilancini et al. 2021).

To account for experimentation, mistakes, or external shocks, we consider a perturbed version of PII. Consistent with the key tenet of PII that transition probabilities depend only on realized payoffs, we consider condition-dependent perturbations (Bilancini and Boncinelli 2020; Bilancini et al. 2020), under which the probability of a perturbation decreases with the current payoff of the updating player. Furthermore, we prove results for the class of generalized condition-dependent perturbations, which includes both standard condition dependence and uniform mistake models (Young 1993a; Kandori et al. 1993) as special cases.[1] Experimental evidence suggests that perturbation probabilities often depend on payoffs (Hwang et al. 2018; Mäs and Nax 2016; Lim and Neary 2016).[2] Therefore, results for classes of dynamics that allow such dependence, such as those in the current paper, are of empirical as well as theoretical importance.

In this paper, we show that the structure of perturbed PII is given by a type of graph—a weak tournament graph, with vertex set equal to the set of strategies and directed edges between vertices, where an edge from one vertex to another indicates that the latter vertex outperforms the former when they interact (Figure 1, left panel). We use this graphical structure to study trajectories

International Economic Review, 2026; 00:1–19
https://doi.org/10.1111/iere.70059

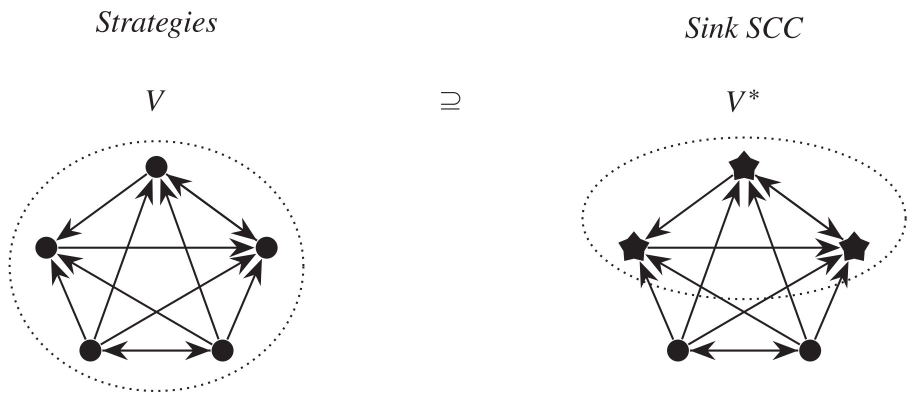

FIGURE 1 | Weak tournament graph on the vertex set of strategies V. The vertex set of the unique sink strongly connected component of the graph is denoted V^*. An arrow from strategy v to v' indicates that in a pairwise interaction between a player who plays v and a player who plays v', the player who plays v obtains a payoff weakly less than the payoff of the player who plays v'.

$$\begin{array}{ccccccccc} \textit{Strategy profiles} & & \textit{Homogeneous profiles} & & \textit{...in a strategy in } V^* & & \textit{Stochastically stable} & & \textit{One-shot stable} \\ S & \supset & \bar{S} & \supseteq & \bar{S}^* & \underset{\text{Lemma 4}}{\supseteq} & SS & \underset{\text{Theorem 1}}{=} & OS(\bar{S}^*) \end{array}$$

FIGURE 2 | Inclusions between sets of strategy profiles. Lemma 4 holds under a mild regularity condition of *competitiveness*. Theorem 1 holds under *generalized condition-dependent perturbations*. When the probability of perturbations decreases in current payoffs, $OS(\bar{S}^*)$ equals the set of efficient strategy profiles within $\bar{S}^*$.

and stable strategy profiles. In particular, tournament graphs have a unique sink strongly connected component (sink SCC), a minimal set of vertices from which there are no exiting edges (Figure 1, star vertices in right panel). It turns out that, under a mild regularity condition, the strategy profiles most likely to be observed in the long run, the stochastically stable strategy profiles (Foster and Young 1990), always have some SCC strategy being played by every player [Lemma 4].

Our main result goes further, showing that, under generalized condition-dependent perturbations, there exists a simple characterization of the set of stochastically stable profiles. Consider the set of strategy profiles at which every player plays the same strategy from the sink SCC, then consider the subset of these profiles at which a single perturbation is least likely to occur. This subset is the one-shot stable set (Newton and Sawa 2015). The one-shot stable set is easy to determine and depends only on local properties of the dynamic. For PII with generalized condition-dependent perturbations, we show that the set of stochastically stable profiles is the one-shot stable set [Theorem 1]. These results are summarized in Figure 2.

We apply our results to a series of examples, including Cournot competition, strategic substitutes versus complements, negative versus positive externalities, the Nash demand game, two strategy games, sexual selection, and status-seeking contests via conspicuous consumption. An underlying principle of PII is the spite effect. Negative externalities make it more likely that you "beat" an opponent, whereas positive externalities make it less likely. As a consequence, relative to best response, PII favors the spread of strategies with negative externalities. For example, in Cournot competition (Section 4.1), V^* contains a single element—the Walrasian competitive quantity, which is greater than the Nash equilibrium quantity.[3] Thus, there is a tendency for strategies in the sink SCC V^* to be less efficient (from the perspective of the players) than Nash equilibria. However, when V^* has multiple elements, for example, the game in Figure 5, this effect is mitigated by condition-dependent perturbations which favor efficient profiles within $\bar{S}^*$ (see Table 2).

Interpreting our results with reference to recent methodological advances, at nonhomogeneous states, a form of asymmetry (see Peski 2010; Newton 2021; Nax and Newton 2022) drives the process toward sink SCC strategies. Any nonsingleton sink SCC V^* has a cyclic structure and a single (generalized) condition-dependent perturbation suffices to move the process between homogeneous states $\bar{S}^*$ associated with these strategies.[4] This leads to the one-shot result $SS = OS(\bar{S}^*)$. Under standard condition dependence, with the probability of perturbations decreasing in payoffs, SS will then consist of the profiles in $\bar{S}^*$ that give the highest payoffs.[5]

1.1 | Related Literature

Bilancini and Boncinelli (2020) study two-strategy Coordination games played under best response with condition-dependent perturbations in populations with random matching. They find that relative stability of conventions depends on (i) the probability of the first perturbation away from a convention (determined by payoffs at the convention) together with (ii) the probability of subsequent perturbations taking the process to the basin of attraction of the alternative convention (determined by disequilibrium payoffs). In contrast, when a one-shot result holds, the probability of the first perturbation alone is typically sufficient to determine stochastic stability. Thus, Bilancini et al. (2020) leveraged the one-shot result for matching problems under perturbed pairwise better response ($SS \subseteq OS$— Newton and Sawa 2015) to construct an evolutionary axiomatization of Rawlsian stable matchings as stochastically stable in every matching problem if and only if perturbations are condition dependent.[6] More recently, building on new results on random games (Johnston et al. 2023), Newton and Sawa (2024) prove a one-shot result for large random normal form games, showing that for almost all normal form games played under perturbed best response, $SS = OS$. Under condition-dependent perturbations, *OS* equals the set of Rawlsian Nash equilibria.

Bilancini et al. (2021) study PII in an infinite population. They say a strategy is *supreme* if it defeats every other strategy in pairwise competition. When a supreme strategy exists, they show almost global asymptotic convergence to that strategy. The equivalent result in the current paper is Lemma 3 for the case of the sink SCC V^* having only one element. They also show and discuss survival of strictly dominated strategies under PII. Campigotto (2021) studies the evolution of conventions in Stag Hunt games. His Proposition 2 shows stochastic stability of the maximin convention under PII with uniform perturbations. Our example in Figure 9a extends this result to PII with any competitive perturbations, as well as showing that Stag Hunts have the same weak tournament graph as other games such as Hawk-Dove and Prisoner's dilemmas.

PII is a form of imitation (see, e.g., Alós-Ferrer and Schlag 2009). In contrast to typical imitative dynamics (Sandholm 2010; Mertikopoulos and Viossat 2022), excess payoff dynamics (Alós-Ferrer and Hofbauer 2022; Sandholm 2005), Riemannian game dynamics (Mertikopoulos and Sandholm 2018), and relative best response (Levine and Pesendorfer 2007), under which switching probabilities depend on overall payoffs (typically averages across random matches), switching probabilities under PII depend on the payoffs from a player's current interaction. To see the difference, consider that a strategy could obtain a very high average payoff, thus doing well under imitative and excess payoff dynamics, but at the same time never obtain a higher payoff than the opponent in any one-on-one interaction, thus doing badly under PII. Clearly, PII is nonmonotone in the sense of Nachbar (1990), which allows strictly dominated strategies to survive (Sethi 1998; Sawa and Zusai 2014; Hofbauer and Sandholm 2011; Kuzmics 2011). Thus, predictions can differ substantially between PII and other imitative dynamics.

Let us consider some models of perturbed imitation in the literature. In general, imitation can be interpersonal or intertemporal (Alós-Ferrer and Shi 2012). In our paper, imitation is interpersonal and there is no payoff averaging. Similarly, Vega-Redondo (1997) has firms in a Cournot game imitate the strategy giving the highest profit, effectively a multiplayer analog of PII without random matching. Our restriction to pairwise interactions allows us to develop a different set of theoretical results, but we see in Section 4.1 that the underlying logic of Vega-Redondo (1997), that spite effects in imitation lead to Walrasian behavior, still holds.

Now consider players who are randomly matched to play a game and compare average payoffs across strategies. In symmetric games, if the payoffs of strategies matched to themselves are given sufficient weight, then imitation can work toward efficiency (Bergin and Bernhardt 2009; Alós-Ferrer 2004; Robson and Vega-Redondo 1996).[7] In contrast, Josephson and Matros (2004) consider asymmetric games where players sample the past and play converges to homogeneous states. Any strategy that is not a best reply can be easily invaded by mutants, so they find that the search for stochastically stable states can be restricted to minimal sets of strategies that are closed under single better replies (analogous to sink SCC sets in the tournament graphs of the current paper).[8,9]

Theoretical and experimental work supports imitative learning rules. Theoretically, Schlag (1998;1999) shows optimality of payoff-dependent imitation, Schipper (2009) shows that imitators outperform myopic optimizers in Cournot games, Duersch et al. (2012) show that the "imitate-if-better" rule (very similar to PII) is robust in that, for many classes of games, it cannot be unboundedly exploited by strategic opponents. In contrast, Hehenkamp and Kaarbøe (2008) show that when payoffs change over time, strategic players can earn higher payoffs than imitators. Experimentally, in studies of Cournot games, Huck et al. (1999), Offerman et al. (2002), Apesteguia et al. (2007), and Bigoni and Fort (2013) show that players tend to imitate when others' actions and payoffs are observable. This is consistent with our model, in which players observe their opponent's action and payoff.

This paper is organized as follows. Section 2 describes the model and gives some preliminary results. Section 3 gives our results on PII and generalized condition dependence. Section 4 studies applications, and Section 5 discusses extensions. Section 6 concludes.

2 | Model

In our model, members of a population meet and play symmetric two-player games against one another. Depending on the payoffs obtained in these games, players adjust their strategies and behavior within the population evolves.

2.1 | Population and Games

Consider a finite population of players $N = \{1, \ldots, n\}$, $n \geq 2$, each of which has a finite set of strategies S_i, $i \in N$. Let $S = \times_{i \in N} S_i$ denote the set of (pure) strategy profiles and $\bar{S} = \{s \in S : \forall\, i, j \in N,\ s_i = s_j\} \subset S$ denote the set of all homogeneous (pure) strategy profiles.

Players will be matched to play a symmetric two-player game. Strategy sets are identical, $S_i = V$ for all $i \in N$, and the payoff function is $u : V^2 \to \mathbb{R}$. That is, if players i, j play $s_i, s_j \in V$, respectively, then i obtains a payoff of $u(s_i, s_j)$.

2.2 | Dynamics and Behavioral Rules

Play will evolve according to a discrete-time Markov process on S. The structure of such a process is very simple: each period, one player will be randomly chosen to update before updating his strategy. However, we want to capture not just a baseline model of behavior, but also perturbed variants of it. With this in mind, we define not just one process, but a family of Markov processes $P = \{P^\varepsilon\}_\varepsilon$ indexed by $\varepsilon \in [0, 1)$, where higher values of ε correspond to a greater frequency of perturbations from the **unperturbed process** P^0. The unperturbed process can be thought of as a baseline model of behavior, from which players occasionally deviate.

For any given level of perturbations ε, the process P^ε is built from the behavior of individual players in the following way (see Figure 3). Let the state at time t be $s^t \in S$. At time $t + 1$, select

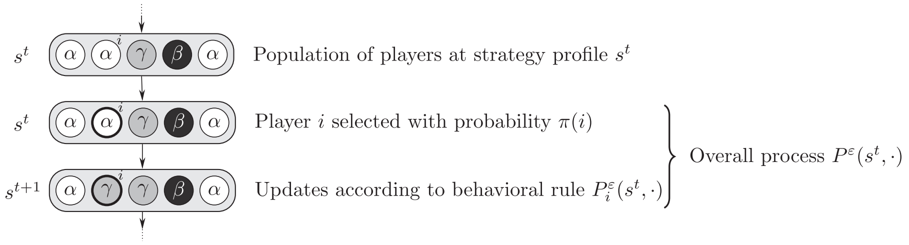

FIGURE 3 | Dynamic process of strategy updating. In a finite population of players, each player is playing a strategy at time t. Here, the strategies are α, β, γ. A player is selected to update according to probability measure π. Here, player i is selected, which occurs with probability $\pi(i)$. Player i then updates his strategy according to his behavioral rule $P_i^\varepsilon(s^t,\cdot)$. Here, he switches to γ, which occurs with probability $P_i^\varepsilon(s^t,(\gamma,s_{-i}^t))$. Details of the behavioral rules we consider are given later.

an updating player $i \in N$ according to a probability measure π with full support on N. That is, $\pi(i)$ is the probability with which player $i \in N$ is selected to update. Following the selection of some player i, let s^{t+1} be randomly determined according to a probability measure $P_i^\varepsilon(s^t,\cdot)$ satisfying $P_i^\varepsilon(s^t,s)=0$ if $s_{-i} \neq s_{-i}^t$. This last constraint says that an updating player i can only change his own strategy s_i, and cannot change the strategies of other players s_{-i}.

Note that P_i^ε is also a Markov process on S. Adopting the terminology of Newton (2021), we refer to the family $P_i = \{P_i^\varepsilon\}_\varepsilon$ as a **behavioral rule** for i. In summary, the two-step strategy updating process selects an updating player before (possibly) updating his strategy, leaving the strategies of other players unchanged. The relationship between P^ε and $\{P_i^\varepsilon\}_{i\in N}$, that is, the relationship between overall processes and individual behavioral rules, is given by

$$P^\varepsilon(s,\cdot) = \sum_{i\in N} \pi(i)\, P_i^\varepsilon(s,\cdot). \tag{1}$$

2.3 | Pairwise Interact and Imitate

The unperturbed process follows the **PII** behavioral rule of Bilancini et al. (2021). The rule is based on imitating successful opponents. The updating player i is randomly matched to a player j. They play the game with their current strategies. If player j obtains a weakly higher payoff than player i in this interaction, then with some probability player i adopts the strategy of player j (see Figure 4a),

$$P_i^0(s,(s_i',s_{-i})) > 0 \iff \begin{array}{c} \exists j \in N :\ s_j = s_i', \\ \text{and} \quad u(s_i,s_i') \le u(s_i',s_i). \end{array} \tag{2}$$

PII models the spread of strategies by individuals who adapt their behavior according to the observed success of their peers in direct interactions. It is a useful model for situations in which doing better than your opponent matters, perhaps due to status concerns or access to some resource that correlates with success in head-to-head contests. The individual information requirements of this dynamic are low. A player only needs to be able to recognize with nonzero probability (i) situations in which he "loses" to an opponent, and (ii) the opponent's strategy in such a situation.

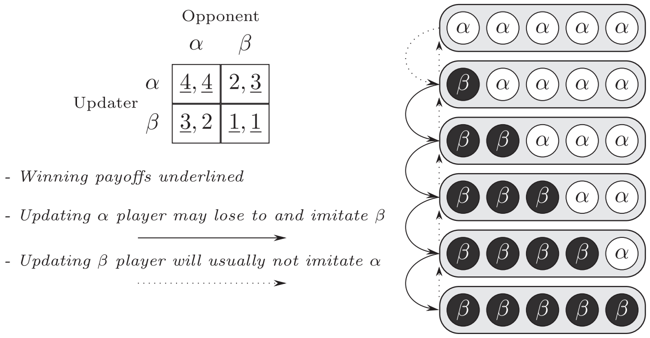

FIGURE 4 | Illustration of PII transitions and perturbations. **(a) Transitions under PII**. For the two-player game on the left, strategy profiles are shown on the right. When an α player plays against a β player, the β player obtains the higher payoff (3 vs. 2). Thus, from heterogeneous states at which some players play α and others play β, PII increases the number of β players. This is despite β being strictly dominated, as it outperforms α in pairwise interactions. **(b) Perturbations**. To move in the opposite direction, or to escape a homogeneous profile (all-α or all-β), requires non-PII transitions that occur with low probability. In this example, even if a perturbation takes the process out of the homogeneous profile all-β, it will usually quickly reenter it. In contrast, from all-α, a single perturbation is enough for β to take over the population with high probability. **(c) Uniform perturbations**. It follows that if all perturbations have probability of order ε, then transition paths from all-α to all-β have much higher probability than transition paths in the opposite direction. **(d) Nonuniform perturbations**. Now, let the first transition away from all-α occur with a much lower probability of order ε^4. This could reflect high payoffs from Coordination on α inducing a reluctance to leave. Even in this case, it is still easier to transition from all-α to all-β than in the opposite direction, as the latter requires five order ε events for an overall probability of order ε^5. This illustrates a general result from later in the paper (Lemma 4): in large enough populations, perturbations never overcome the underlying forces of PII, but can distinguish between outcomes when PII gives no clear winner (Theorem 1). For example, if we amend the current example so that the payoffs from (α,β) are (2,2), then a single perturbation would suffice to move in either direction between homogeneous states. Which of these two transition paths is easier would then depend on perturbation probabilities.

It follows from the definition of PII that, under the unperturbed process, as long as multiple strategies are played, there is always a positive probability of some player changing strategy. Eventually, a homogeneous strategy profile is reached (see Figure 4a).

Lemma 1. *From any initial state, the unperturbed process* P^0 *will reach a homogeneous absorbing state* $s \in \bar{S}$ *in finite time with probability one.*

Convergence to homogeneity is a common feature under imitative processes. Examples of similar results can be found in Vega-Redondo (1997, Proposition 1), Josephson and Matros (2004, Theorem 1), and Bergin and Bernhardt (2009, proof of Theorem 1). Finally, observe that the weak inequality between payoffs on the right-hand side of (2) ensures that the process cannot get stuck at heterogeneous states when two strategies played against one another result in tied payoffs. Further discussion of this and other aspects of model robustness can be found in Section 5.

2.4 | Perturbed Process and Stochastic Stability

Strategy updating in real-world situations is not always precise. A player who typically follows PII may occasionally, through mistake, mutation, or experimentation, deviate from PII and update his strategy in some other manner. The behavioral rule $P_i = \{P_i^\varepsilon\}_\varepsilon$ can be used to describe such phenomena, with the magnitude of ε describing how likely it is that a perturbation from PII occurs. Given this motivation, we adopt the natural assumption that P_i^ε is continuous in ε.

To simplify exposition, we allow our perturbed behavioral rules to choose any strategy. Considering player i, every $s_i \in S_i$ is selected with positive probability.

$$\text{For all } \varepsilon > 0,\ s \in S,\ s'_i \in S_i, \quad P_i^\varepsilon(s, (s'_i, s_{-i})) > 0. \tag{3}$$

Although (3) implies that all probabilities are positive when $\varepsilon > 0$, continuity of P_i^ε implies perturbations from PII occur with vanishing probability as $\varepsilon \to 0$ and $P_i^\varepsilon \to P_i^0$. Assumption (3) further implies that, for $\varepsilon > 0$, P^ε is irreducible and has a unique invariant probability measure μ^ε on the state space S.

It is important to note that (3) does not imply that all perturbations become equally likely as we consider sequences of strictly positive ε that approach 0. Indeed, it is possible that some perturbations become infinitely more likely than others. This becomes clear with our next assumption, that behavioral rules are **regular** (Young 1993a; see also Sandholm 2010). Regularity means that any transition probability that vanishes as $\varepsilon \to 0$ is of the order of magnitude of ε^k for some $k > 0$. This k can differ according to the transition under consideration.[10] The larger is this k, the less probable is the transition when ε is small. Figure 4b–d illustrates examples of perturbations with both uniform and varying probabilities.

Standard arguments (see Sandholm 2010) imply that the limit of μ^ε as $\varepsilon \to 0$ exists. Denote this limit by μ^0. For small ε, the process will spend most of the time at states which have positive probability under this limiting measure. These are known as **stochastically stable** states (Foster and Young 1990).

$$SS := \left\{ s \in S \,:\, \lim_{\varepsilon \to 0} \mu^\varepsilon(s) = \mu^0(s) > 0 \right\}. \tag{5}$$

Note that stochastic stability is a globally determined property, arising from all of the transition probabilities in the process. This contrasts with the property we next consider, one-shot stability.

2.5 | Cost Functions and One-Shot Deviation

Define the **cost** $c(s, s')$ of a transition from s to s' as the exponential rate of decay of the probability of such a transition as $\varepsilon \to 0$.

$$c(s, s') := \lim_{\varepsilon \to 0} \frac{\log P^\varepsilon(s, s')}{\log \varepsilon}. \tag{6}$$

Cost functions measure the order of magnitude of transition probabilities for low values of ε. Transitions with high cost are less likely than transitions with low cost.

- If a transition can occur under the unperturbed process P^0, then $c(s, s') = 0$. For example, player i imitating j and switching from s_i to s_j when $u(s_i, s_j) \leq u(s_j, s_i)$.
- If a transition is only possible for $\varepsilon > 0$, then $c(s, s') = k$, where k is the k from (4). For example, player i switching from s_i to s_j even though $u(s_i, s_j) > u(s_j, s_i)$.
- Finally, if a transition from s to s' is impossible, we have $c(s, s') = \infty$. For example, when multiple players' strategies differ between s and s'.

Define the **least cost transition from strategy profile** $s \in S$ as solving

$$c(s) := \min_{s' \in S \setminus \{s\}} c(s, s'). \tag{7}$$

Thus, $c(s)$ measures how unlikely is the most probable departure from strategy profile s. If $c(s)$ is a low number, departures from s are likely. If $c(s)$ is a high number, departures from s are unlikely.

Denote the set of states in $T \subseteq S$ that are most resilient to a single deviation, the set of **one-shot stable** (Newton and Sawa 2015) states in T, by

$$OS(T) := \operatorname*{argmax}_{s \in T} c(s). \tag{8}$$

Under our unperturbed process, if s is not homogeneous (i.e., $s \notin \bar{S}$), then there exists a transition in which a player imitates an opponent with a different strategy. Therefore, $c(s) = 0$. Conversely, if s is homogeneous (i.e., $s \in \bar{S}$), then there is no transition to $s' \neq s$ that occurs under the unperturbed process, therefore, $c(s) > 0$. Substituting into the definition of $OS(T)$, it follows that if $T \cap \bar{S} \neq \emptyset$, then $OS(T) \subseteq T \cap \bar{S}$. That is, whenever possible, one-shot stable states will be homogeneous.

Observe that $OS(\cdot)$ is determined by the local properties of the dynamics at each state. Specifically, if it is easier to escape s than to escape s', then $OS(\cdot)$ will never select s when s' is an option. What occurs at other states is irrelevant to this comparison. This contrasts with SS, which is a globally determined property.

TABLE 1 | Summary of mathematical notation used in the paper.

Notation	Object
N, n	Player set of size n
$i \in N$	Player
$s_i \in S_i$	Strategies for player i (identical across players)
$s \in S$	Strategy profiles
$\bar{S}$	Homogeneous strategy profiles
$u(\cdot, \cdot)$	Payoff function
ε	Perturbation parameter
P^{ε}	Strategy updating process
$\pi(i)$	Updating probability for player i
P_i^{ε}	Behavioral rule for player i
μ^{ε}	For $\varepsilon > 0$, unique stationary distribution of P^{ε}
μ^0	Limit of μ^{ε} as $\varepsilon \to 0$
$c(s, s')$	Cost of a transition from s to s'
$c(s)$	Least cost transition from s
$OS(T)$	One-shot stable strategy profiles in $T \subseteq S$
SS	Stochastically stable strategy profiles
$G = (V, E)$	Directed graph with vertex set V and edge set E
R	Reachability graph edge set
V^*	Sink strongly connected components of G
$\bar{S}^*$	Homogeneous profiles in a strategy in V^*

2.6 | Weak Tournament Graphs

We complete the model section by describing a class of graphs, tournament graphs, that play an important role in the analysis. Essentially, the structure of our unperturbed dynamic of pairwise interaction and imitation can be represented graphically. For this, some definitions are necessary.

Let $G = (V, E)$ be a directed graph with vertex set equal to the set of strategies V. Edge set E is a set of ordered pairs $(v, v') \in V \times V$, $v \neq v'$.

1. G is a **weak tournament graph** if, for all $v, v' \in V$, $v \neq v'$, the edge set E contains at least one of (v, v'), (v', v).
2. A **path** from v to v' in G is a finite sequence of edges $(v^k, v^{k+1}) \in E$, $k = 0, \dots, K-1$ such that $v^0 = v$ and $v^K = v'$.
3. An **SCC** of G is a maximal subset $V' \subseteq V$ such that a path exists from any $v \in V'$ to any $v' \in V'$.
4. A **sink SCC** of G is an SCC $V^* \subseteq V$ such that $v \in V^*$, $v' \notin V^*$ implies that $(v, v') \notin E$.

Table 1 summarizes the mathematical notation. Tournament graphs capture the structure of outcomes from pairwise competition. For example, a tournament graph might illustrate who beats whom in a boxing competition. A path then represents of a sequence of wins and a sink SCC a minimal set of boxers such that no boxer outside of the set defeats any boxer in the set.[11] A set of champions, so to speak. In our model, the tournament will not be between boxers, but between strategies, and winning will be equivalent to the possibility of being imitated. We now proceed to analysis that will make this relation precise.

3 | Analysis

Results are presented in two parts. First, we give results on the unperturbed process. Second, we give results on the perturbed process. In each case, we consider the structure and convergence of behavior under the model.

3.1 | The Unperturbed Process

We begin by connecting the unperturbed process to a specific tournament graph. Let $G = (V, E)$ be the directed graph with vertex set equal to the set of strategies V and an edge from v to v' if and only if playing v' against v yields a payoff at least as great as that from playing v against v'. That is,

$$E = \{(v, v') : \ v \neq v', \, u(v, v') \leq u(v', v)\}. \tag{9}$$

Note that E includes at least one directed edge between any pair of vertices in V. Therefore, $G = (V, E)$ is a weak tournament graph. Moreover, this property that it is always possible to go from v to v' or from v' to v implies the existence of a unique sink SCC. If there were two such components V^* and $V^{*\prime}$, with $v \in V^*$ and $v' \in V^{*\prime}$, then it would be possible to move from one component to the other, which would contradict the definition of sink SCC.

Lemma 2. *G is a weak tournament graph. Consequently, G has a unique sink SCC $V^* \subseteq V$.*

Define $\bar{S}^*$ to be the set of homogeneous strategy profiles at which every player plays some strategy $v \in V^*$.

$$\bar{S}^* = \left\{ s = (v, \dots, v) \in \bar{S} : \ v \in V^* \right\}. \tag{10}$$

Strategies $v \in V^*$ have no outgoing edges to any $v' \notin V^*$. The similarity between the definition of the edge set E and the unperturbed dynamic P^0 as described in (2) leads us to the fact that, as long as at least one player in the population plays a strategy in V^*, the unperturbed dynamic will eventually lead to the elimination of all strategies in $V \setminus V^*$.

Lemma 3. *If $s \in S$ is such that $\{i \in N : \ s_i \in V^*\} \neq \emptyset$, then, starting from s, the unperturbed process P^0 will reach some $s' \in \bar{S}^*$ in finite time with probability one.*

Lemma 3 characterizes outcomes of the unperturbed process. If at least one of the strategies from the set of "winners" (the sink SCC V^*) is initially present in the population, then the process converges to a homogeneous profile in such a strategy. PII ensures that strategies in V^* can only be beaten by other strategies in V^*. Thus strategies that are not in V^* gradually go extinct and the homogenizing power of imitation eventually guarantees that only one strategy remains in the population.

Remark 1. Bilancini et al. (2021) say that a strategy $v \in V$ is **supreme** if $u(v, v') > u(v', v)$ for all $v' \in V \setminus \{v\}$. Their Propo-

$$
\begin{array}{cc|c|c|c|}
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 & & \alpha & \beta & \gamma \\
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 & \alpha & 4 & 2 & 2 \\
\hline
s_i & \beta & 3 & 1 & 1 \\
\hline
 & \gamma & 3 & 1 & 0 \\
\hline
\end{array}.
$$

FIGURE 5 | Payoffs of a symmetric game. For example, if $s_i = \alpha$ and $s_j = \beta$, then $u(s_i, s_j) = 2$. Note that α is a strictly-dominant strategy. That is, fixing any s_j, $u(s_i, s_j)$ is uniquely maximized by $s_i = \alpha$.

sition 1 shows almost global convergence in a continuum population to homogeneous play of such a strategy. In our context, supremacy of v implies $V^* = \{v\}$. Thus, when a supreme strategy exists, Lemma 3 is a finite population version of their result.

With a slight abuse of notation, consider the edge set E as a matrix with one row and one column for each strategy, in which an entry of 1 indicates the existence of a directed edge from the row strategy to the column strategy, and an entry of 0 indicates the absence of such an edge. From strategy v, strategy v' is **reachable** if there exists a path in G from v to v'. The reachability graph has edge set $R = I \vee E \vee \cdots \vee E^{|V|-1}$, where $\vee$ denotes the logical "or" operation. Note that the strategies can always be ordered so that R is block upper triangular with 1's above the diagonal. Each block of the matrix corresponds to an SCC, and the bottom right block corresponds to the unique sink SCC.

The reachability graph represents multistep transitions between strategies that could theoretically occur under P^0. Even if $u(v, v') > u(v', v)$, so that a player never switches directly from v to v' under PII, it might still be the case that v' is reachable from v via some indirect path. For example, there might exist v'' such that $u(v, v'') \leq u(v'', v)$ and $u(v'', v') \leq u(v', v'')$, so that a player might switch from v to v'' in one period, then from v'' to v' in a later period.

3.1.1 | Illustrative Example: Strictly-Dominant Strategy Eliminated

Consider the game with strategy set $V = \{\alpha, \beta, \gamma\}$ and payoffs $u(s_i, s_j)$ given by the matrix in Figure 5. This gives the weak tournament graph $G = (V, E)$ with edge set E given by the following matrix (recall that entries denote the presence or absence of a directed edge from the row strategy to the column strategy),

$$
E = \begin{array}{c c}
 & \alpha\ \beta\ \gamma \\
\begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\end{array}
$$

which, in turn, gives the reachability graph with edge set

$$
R = I \vee E \vee E^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.
$$

As noted above, strategies can be ordered so that R is block upper triangular with 1's above the diagonal. In this example, this is already the case,

$$
R = \begin{bmatrix} [1] & \mathbf{1} \\ \mathbf{0} & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{bmatrix}.
$$

The blocks on the diagonal of R correspond to SCCs $\{\alpha\}$ and $\{\beta, \gamma\}$. The SCC in the bottom right corner corresponds to the unique sink SCC $\{\beta, \gamma\}$.

Thus, $V^* = \{\beta, \gamma\}$, so Lemma 3 implies that, under the unperturbed process P^0, from any profile at which at least one player plays β or γ, the set

$$
\bar{S}^* = \{(\beta, \ldots, \beta), (\gamma, \ldots, \gamma)\}
$$

is reached in finite time with probability one. Note that, although players can switch from β to γ and from γ to β, randomness in the process ensures that eventually a homogeneous profile is reached. Heterogeneous profiles do not persist. An overview of transitions for this example is given in Figure 6.

3.2 | The Perturbed Process

We now move to considering the perturbed dynamic. We restrict transition probabilities so that no sequence of $|N| = n$ consecutive perturbations is more likely than any single perturbation. This assumption prevents perturbation probabilities from differing too much from one another. Without it, perturbations away from some strategy might be so unlikely relative to other perturbations as to render the unperturbed process irrelevant to the long-term evolution of the process.

Definition 1. If there exists $\varkappa > 0$ such that $\varkappa \leq c(s, s') < \varkappa n$ for all s, s' with $0 < c(s, s') < \infty$, then we say that the process $P = \{P^\varepsilon\}_\varepsilon$ is **competitive**.

A simple example of a competitive process is one in which every strategy change by player i that cannot occur under P_i^0 occurs with probability $\varepsilon/|S|$ under P^ε. Definition (6) then implies that every perturbation has a cost equal to 1, clearly satisfying the definition of competitiveness. To adjust this example to make the process not competitive, let the probability of perturbations away from one given strategy $v \in S$ instead occur with probability $\varepsilon^{2n}/|S|$ and have a cost of $2n$. Comparing $2n$ to 1, we see that competitiveness is not satisfied.

When P is competitive, the cost of any single perturbation does not exceed the total cost of any n perturbations. Lemma 3 implies that, from any state, at most a single perturbation is required for a transition to $\bar{S}^*$, whereas to move in the opposite direction requires every $i \in N$ to switch to some strategy in $V \setminus V^*$ (i.e., n perturbations are required). Thus, under a competitive P, it is easier for the process to move from $\bar{S} \setminus \bar{S}^*$ to $\bar{S}^*$ than to move in the opposite direction.[12]

Lemma 4. *If P is competitive, then $SS \subseteq \bar{S}^*$.*

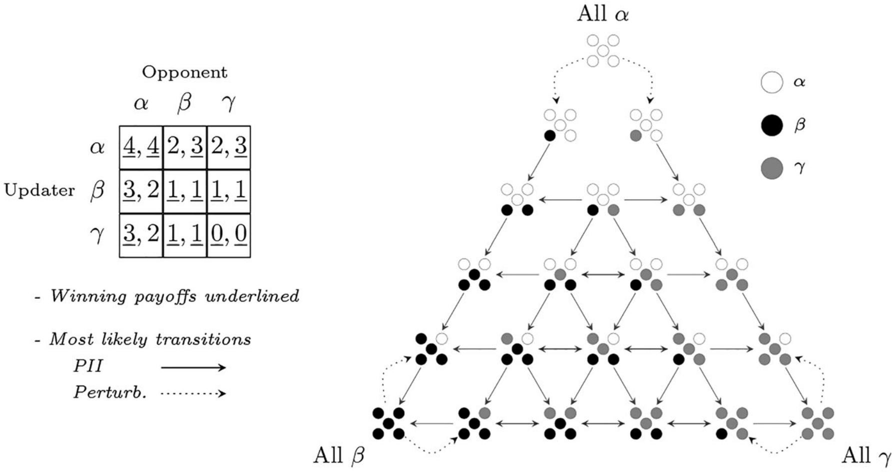


FIGURE 6 | Representative strategy profiles and transitions for the Example in Figure 5 when $n = 5$. (a) PII favors strategies β and γ over α. (b) The relative magnitudes of probabilities of transitions between all-β and all-γ depend on perturbations.

In general, the probability and therefore cost of a transition from s to (s_i', s_{-i}) can depend on both s and s_i'. An interesting special case is when it only depends on s. The classic case of this is condition-dependent perturbations, where players who obtain high payoffs are less likely to make errors in strategy choice (Bilancini and Boncinelli 2020). Here, we are more general, allowing any dependence on s.

Definition 2. If behavioral rule P_i is such that $c(s, (s_i', s_{-i})) \in \{0, d_i(s)\}$ for some function $d_i : S \to \mathbb{R}_{++}$, then we say that P_i is **generalized condition dependent**. If this holds for all behavioral rules P_i, $i \in N$, we say that process $P = \{P_i\}_{i \in N}$ is generalized condition-dependent.

A simple example of a generalized condition-dependent behavioral rule is our unperturbed dynamic together with uniform perturbations. Strategy updates that cannot occur under the unperturbed dynamic all have the same cost.

Example1 (Uniform perturbations). If $d_i(s) = \delta_i$ for some constant $\delta_i > 0$, then P_i is **uniform**.

Another example of generalized condition dependence is the standard approach to condition dependence, under which higher current payoffs lead to lower perturbation probabilities. If player i is matched to another player and obtains a low payoff, then dissatisfaction leads to a higher probability (lower cost) that player i will play a strategy that is not possible under the unperturbed process.

Example2 (u-condition dependence). Let $U_i(s) = \{v \in V : s_j = v \text{ for some } j \neq i\} \subseteq V$, so that $U_i(s)$ is the set of strategies that are played at profile s by some player other than player i. If $d_i(s) = f(\min_{v \in U_i(s)} u(s_i, v))$ for some strictly increasing function $f : \mathbb{R} \to \mathbb{R}_{++}$, then P_i is **u-condition dependent**. The probability of a perturbation by player i decreases in the worst possible payoff that could be obtained from playing against another player.

Example3 (Homogeneous condition dependence). If, for all $i \in N$, $d_i(s) = d(s)$, then P is **homogeneous**.

Perturbations excluded by generalized condition dependence are those in which perturbed transition probabilites from s to s' depend not only on the current profile s but also on the destination profile s'. An example of such a nongeneralized condition-dependent perturbation is logit choice, where the probability of player i switching from s_i to s_i' decreases in the payoff loss that player i incurs from such a switch.

3.3 | Main Result

We now give the main result of the paper, the equivalence, under generalized condition-dependent perturbations, of stochastically stable profiles SS and profiles that are one-shot stable $OS(\bar{S}^*)$ among all homogeneous profiles in sink SCC strategies $\bar{S}^*$.

Lemma 4 ensures that $SS \subseteq \bar{S}^*$. The correspondence between $\bar{S}^*$ and the sink SCC of G then allows us to show that, under any generalized condition-dependent process, for any two distinct $s, s' \in \bar{S}^*$ there exists a path from s to s' that uses only least cost transitions. This leads to the following result.

Theorem 1. *If P is competitive and generalized condition dependent, then $SS = OS(\bar{S}^*)$.*

The intuition behind Theorem 1 is illustrated in Figure 7. Lemma 4 tells us that PII favors transitions toward states in $\bar{S}^*$. Selection within $\bar{S}^*$ is then driven by the probabilities of transitions between profiles in $\bar{S}^*$. Under generalized condition dependence and PII, the most probable such transitions can be characterized as follows. From $s = (v, \ldots, v) \in \bar{S}^*$, a single player switches to v', where $v \to v'$ is an edge in the tournament graph (G, E). This implies that PII can then move the process to $s' = (v', \ldots, v')$ without any further perturbations. As V^* is strongly connected, by a series of such transition paths it is possible to move from any $s \in \bar{S}^*$ to any target $s' \in \bar{S}^*$. By considering trees composed of unions of such paths, we arrive at a rate at which probability flows into any $s' \in \bar{S}^*$. These rates can then be compared. As initial perturbations from profiles in $OS(\bar{S}^*) \subseteq \bar{S}^*$ are less probable than those from other profiles in $\bar{S}^*$, probability accumulates on $OS(\bar{S}^*)$. That is, these states are stochastically stable.

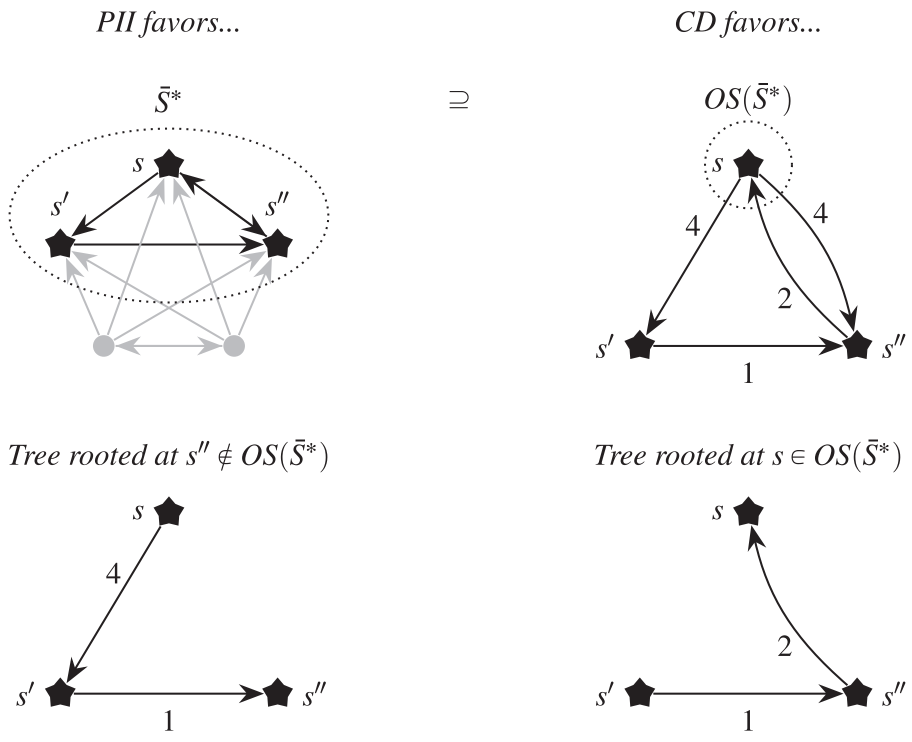


FIGURE 7 | Consider our illustration from Figure 1, but identify each node (i.e., strategy) with the homogeneous profile in that strategy. Lemma 4 tells us that $SS \subseteq \bar{S}^*$ [**top left, where arrows represent the tournament graph as in** Figure 1]. Theorem 1 tells us that $SS \subseteq OS(\bar{S}^*)$ [**top right, where arrows give the most probable transitions away from each homogeneous profile, labeled with the cost of that transition**]. The intuition for Theorem 1 is that the flow of probability into any given state can be measured by the minimal sum of transition costs over trees that include paths from every state into the state in question. This is known in the literature as *stochastic potential* (see Young 1998). Without loss of generality, we can restrict attention to $\bar{S}^*$. Note **[bottom left]** that trees rooted outside $OS(\bar{S}^*)$ have sums of costs [here $4 + 1 = 5$] greater than the sum of costs of $1 + 2 = 3$ of the tree rooted in $OS(\bar{S}^*)$ **[bottom right]**. Thus, it is easier to get to the latter state, making it stochastically stable.

The role of condition dependence can be seen in our above construction of paths from s to s', $(v, v') \in E$. Under generalized condition dependence, there is a least cost transition path from s to s' in which a player switches from v to v' and the remaining players switch to v' under P^0. We know this is a least cost path as it combines a least cost transition from s with subsequent zero cost transitions. Generalized condition dependence ensures that the initial switch from v to v' is a least cost transition. Without this assumption, all least cost transition paths from s to s' might be indirect, involving intermediate transitions to strategies other than v and v'.

If the sink SCC equals the set of all vertices, $V^* = V$, the tournament is known as a strongly connected tournament. In this case, as $\bar{S}^* = \bar{S}$, selection by one-shot stability extends to the entire state space.

Corollary 1. *If* $V^* = V$, *then* $SS = OS(\bar{S}) = OS(S)$.

Remark 2. If, for all $v, v' \in V$, we let $(v, v') \in E$ with probability 1/2, otherwise we let $(v', v) \in E$, then the probability $Q(V)$ that $V^* \neq V$ is bounded (see Moon and Moser 1962, Theorem 1),

$$Q(V) < \frac{1 + 2|V|}{2^{|V|-1}} \quad \text{for} \quad |V| \geq 14. \tag{12}$$

This random assignments of edges can be achieved if we let $\{u(v, v')\}_{v,v' \in V}$ be independent identical distributed continuous random variables. Under these random payoffs, the probability that $u(v, v') > u(v', v)$ and the probability that $u(v, v') < u(v', v)$ are both equal to 1/2, as desired.

If we remove the requirement that the iid distribution be continuous, then we obtain the possibility of ties. The probability that $u(v, v') > u(v', v)$ still equals the probability that $u(v, v') < u(v', v)$, but we now have the additional possibility that $u(v, v') = u(v', v)$, in which case both (v, v') and (v', v) are in E. However, by definition of sink SCC, having both of these edges in E rather than just one of them does not reduce the size of V^*. Consequently, the bound (12) still holds, so as $|V| \to \infty$, the conditions of Corollary 1 hold asymptotically almost surely. ■

Let us apply Theorem 1 to the behavioral rules described in Examples 1 to 3. If all players have the same uniform behavioral rule, then Theorem 1 implies that any homogeneous strategy profile at which a sink SCC is played is stochastically stable. Note that homogeneity and uniformity together imply that P is competitive.

Corollary 2. *Let* P *be homogeneous and uniform. Then* $SS = \bar{S}^*$.

If all players have the same u-condition-dependent behavioral rule, then Theorem 1 implies that stochastically stable states are the homogeneous profiles of sink SCCs that maximize players' payoffs.

Corollary 3. *Let P be competitive, homogeneous, and u-condition dependent. Then, Theorem 1 and the definition of $OS(\cdot)$ implies that $SS = \underset{s=(v,...,v)\in\bar{S}^*}{\operatorname{argmax}} \; u(v,v)$.*

3.3.1 | Illustrative Example (Continued)

Consider the example from Section 3.1.1. Lemma 4 implies that, if P is competitive, then stochastically stable states SS satisfy

$$SS \subseteq \bar{S}^* = \{(\beta,\ldots,\beta),(\gamma,\ldots,\gamma)\}.$$

When P is homogeneous and uniform, we have

$$\text{For all } s \in \bar{S}^*, \qquad c(s) = d(s) = \delta, \tag{13}$$

where the first equality follows from homogeneity and the second from uniformity. From (13) and (8), the set of one-shot stable states in $\bar{S}^*$ is

$$OS(\bar{S}^*) = \bar{S}^*.$$

Consequently, if P is competitive, then Theorem 1 tells us that $SS = OS(\bar{S}^*) = \bar{S}^*$, consistent with the statement of Corollary 2.

Intuitively, PII will usually quickly take the process to one of $(\beta,\ldots,\beta)$ or $(\gamma,\ldots,\gamma)$. Each is absorbing under P^0, so a perturbation is necessary to exit the state and enter some heterogeneous state. From a heterogeneous state that includes both β and γ, either of $(\beta,\ldots,\beta)$ and $(\gamma,\ldots,\gamma)$ can be reached under PII. Therefore, the only perturbations to consider on the most likely transition paths between these states are the initial perturbations. By homogeneity and uniformity, any perturbation from either of these states has probability of similar order of magnitude. Therefore, transitions from $(\beta,\ldots,\beta)$ to $(\gamma,\ldots,\gamma)$ occur approximately as often as transitions in the opposite direction. The process spends similar amounts of time at each of these states. Both are stochastically stable.

Now consider P that is homogeneous and u-condition dependent, with $f(4)/f(0) < n$ to ensure competitiveness. For $s = (v',\ldots,v') \in \bar{S}^*$ and arbitrary $i \in N$, we have

$$c(s) = d_i(s) = f\left(\min_{v\in U_i(s)} u(s_i,v)\right) \underbrace{=}_{\substack{\text{by } s_i=v' \\ \text{and } U_i(s)=\{v'\}}} f(u(v',v')), \tag{14}$$

from which we obtain

$$\begin{aligned} c((\beta,\ldots,\beta)) &= f(u(\beta,\beta)) = f(1) \\ &> f(0) = f(u(\gamma,\gamma)) = c((\gamma,\ldots,\gamma)). \end{aligned}$$

Together with (8), this implies that the set of one-shot stable states in $\bar{S}^*$ is

$$OS(\bar{S}^*) = \{(\beta,\ldots,\beta)\}.$$

Consequently, if P is competitive, then Theorem 1 tells us that $SS = OS(\bar{S}^*) = \{(\beta,\ldots,\beta)\}$, consistent with the statement of Corollary 3.

Intuitively, from $(\beta,\ldots,\beta)$, a switch by some player to γ has probability of order of magnitude $\varepsilon^{f(1)}$. From $(\gamma,\ldots,\gamma)$, a switch by some player to β has probability of order of magnitude $\varepsilon^{f(0)}$. For small ε, the latter probability is considerably larger than the former, so transitions from $(\gamma,\ldots,\gamma)$ to $(\beta,\ldots,\beta)$ are much more likely than transitions in the opposite direction. Therefore, the process spends most of its time at $(\beta,\ldots,\beta)$, which is uniquely stochastically stable.

4 | Applications

In this section, we consider our model and dynamics in several applications. These are Cournot competition; a general formulation of complements, substitutes, and externalities; the Nash demand game; two strategy games; sexual/social selection; and contests and conspicuous consumption.

4.1 | Cournot Competition

Cournot competition is a canonical model of oligopoly (Cournot 1838). It provides a natural setting in which strategic interaction is both tractable and economically meaningful. By increasing production, a firm takes into account that it lowers the market price for its own production, but does not consider the negative externality that it imposes on the other firm. It has been found that, under some imitative dynamics (see Vega-Redondo 1997; Alós-Ferrer 2004; Alós-Ferrer and Shi 2012), there is tendency for selection in favor of higher production quantities than those predicted by Nash equilibrium. Indeed, a tendency toward Walrasian equilibrium. We find a similar effect under our model.

Let $c > 0$ be a constant marginal cost of production, $p(\cdot)$ be a strictly decreasing function that gives price as a function of aggregate quantity. Let $p(0) > p(1) > \ldots > p(Q^*) = c$, so that Q^* is the aggregate quantity at which the price reaches c. For simplicity, assume that $Q^*/2$ is an integer. Consider the game with strategy set $V = \{0,1,\ldots,Q^*\}$ and payoffs $u(s_i,s_j) = \big(p(s_i+s_j) - c\big)\, s_i$. That is, each player chooses a production quantity, then these quantities determine price and profits.

More production is as good as less production if and only if prices are weakly positive, $p(\cdot) \geq 0$. Similarly, less production is as good as more production if and only if prices are weakly negative, $p(\cdot) \leq 0$.

It follows that, if $s_i < Q^*/2$, then $u(s_i,s_j) \leq u(s_j,s_i)$ if and only if $s_i \leq s_j \leq Q^* - s_i$. Similarly, if $s_i > Q^*/2$, then $u(s_i,s_j) \leq u(s_j,s_i)$ if and only if $Q^* - s_i \leq s_j \leq s_i$. Finally, if $s_i = Q^*/2$, then $u(s_i,s_j) \leq u(s_j,s_i)$ if and only if $s_j = Q^*/2 = s_i$. Therefore, the weak tournament graph $G = (V,E)$ is given by

$$E = \begin{array}{c|ccccccccccc|}
 & 0 & 1 & 2 & \cdots & \frac{Q^*}{2}-1 & \frac{Q^*}{2} & \frac{Q^*}{2}+1 & \cdots & Q^*-2 & Q^*-1 & Q^* \\
0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\
2 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{Q^*}{2}-1 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\
\frac{Q^*}{2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\frac{Q^*}{2}+1 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
Q^*-2 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\
Q^*-1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\
Q^* & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 0
\end{array}.$$

Reordering the strategies, we obtain

$$E = \begin{array}{c} \\ Q^* \\ 0 \\ Q^*-1 \\ 1 \\ \vdots \\ \frac{Q^*}{2}+1 \\ \frac{Q^*}{2}-1 \\ \frac{Q^*}{2} \end{array} \begin{array}{c} \begin{array}{cccccccc} Q^* & 0 & Q^*-1 & 1 & \cdots & \frac{Q^*}{2}+1 & \frac{Q^*}{2}-1 & \frac{Q^*}{2} \end{array} \\ \left[\begin{array}{cccccccc} 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right] \end{array},$$

which gives the reachability graph $R = I \vee E \vee \ldots \vee E^{|V|-1}$,

$$R = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & & & \\ & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \mathbf{1} & \\ & & \ddots & \\ & \mathbf{0} & & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \\ & & & & \begin{bmatrix} 1 \end{bmatrix} \end{bmatrix}.$$

Therefore, SCCs are the pairs $\{s_i, Q^* - s_i\}$, $s_i = 0, \ldots, Q^*/2 - 1$, as well as the unique sink SCC $\{Q^*/2\}$. Consequently, Lemma 3 implies that, under the unperturbed process P^0, from any profile at which at least one player plays $Q^*/2$, the profile $(Q^*/2, \ldots, Q^*/2)$ is reached in finite time with probability one. Lemma 4 implies that, if P is competitive, then $SS = \{(Q^*/2, \ldots, Q^*/2)\}$. That is, when the game is played between two players, aggregate production equals $Q^*/2 + Q^*/2 = Q^*$, giving a price $p(Q^*) = c$. That is, the unique stochastically stable state is the symmetric Walrasian equilibrium. This is similar to the results of Vega-Redondo (1997). In both cases, relative payoffs determine imitative behavior, so that spite effects drive profits to zero.

4.2 | Complements, Substitutes, and Externalities

A standard classification of dependencies between players' incentives is whether players' actions are strategic complements or substitutes. Similarly, dependencies between a player's actions and the payoffs of other players can be classified by whether the externalities generated are positive or negative. These two dimensions are important determinants of the structure of equilibria and the efficiency of outcomes. In this section, we consider all four cases: substitutes versus complements, positive versus negative externalities, within a simple framework. By doing so, we place the Cournot model of Section 4.1 within a broader taxonomy, highlight where our dynamics align with welfare maximization, and identify systematic sources of inefficiency.

Consider $V = \{0, 1, \ldots, Q^*\}$ as in Section 4.1. Consider the payoff functions given in Figure 8a. These are divided along the dimensions of strategic substitutes versus complements and negative versus positive externalities.

Externalities \ Strategic...	**Substitutes**	**Complements**
Negative	$(Q^* - s_i - s_j)s_i$	$(Q^* - s_i)(s_i - s_j)$
Positive	$(Q^* - s_i)(s_i + s_j)$	$(Q^* - s_i + s_j)s_i$

(a) $u(s_i, s_j)$.

Externalities \ Strategic...	**Substitutes**	**Complements**
Negative	$(s_i - s_j)(Q^* - s_i - s_j)$	$(s_i - s_j)(2Q^* - s_i - s_j)$
Positive	$(s_i - s_j)(s_i + s_j)$	$(s_i - s_j)(Q^* - s_i - s_j)$

(b) $u(s_i, s_j) - u(s_j, s_i)$.

FIGURE 8 | Payoff functions $u(s_i, s_j)$ and payoff comparisons $u(s_i, s_j) - u(s_j, s_i)$ for strategic substitutes, versus complements and negative versus positive externalities.

The payoff function in Section 4.1 was $u(s_i, s_j) = \big(p(s_i + s_j) - c\big)\, s_i$, so that

$$u(s_i, s_j) - u(s_j, s_i) = (s_i - s_j)\big(p(s_i + s_j) - c\big),$$

which for the special case $p(s_i + s_j) = Q^* - (s_i + s_j) + c$ reduces to

$$u(s_i, s_j) - u(s_j, s_i) = (s_i - s_j)\big(Q^* - s_i - s_j\big),$$

which is identical to the cases in Figure 8b of **strategic substitutes with negative externalities** and **strategic complements with positive externalities**. Therefore, the analysis of stability follows Section 4.1. Note, however, that the welfare-maximizing homogeneous strategy profile differs across these cases. With strategic substitutes and negative externalities, the welfare-maximizing strategy is $Q^*/4$, so that the strategy $Q^*/2$ favored by our processes is inefficiently high. With strategic complements and positive externalities, the welfare-maximizing strategy is Q^*, so that the strategy $Q^*/2$ favored by our processes is inefficiently low.

Consider strategic complements and negative externalities. If follows from Figure 8b that $u(s_i, s_j) \geq u(s_j, s_i)$ if and only if $s_i \geq s_j$. Consequently, ordering strategies $0, 1, \ldots, Q^*$, the reachability graph R has blocks that consist of a single element. Therefore, SCCs are the singletons $\{s_i\}$, $s_i = 0, \ldots, Q^*$, with unique sink SCC $\{Q^*\}$. Consequently, Lemma 3 implies that, under the unperturbed process P^0, from any profile at which at least one player plays Q^*, the profile $(Q^*, \ldots, Q^*)$ is reached in finite time with probability one. Lemma 4 implies that, if P is competitive, then $SS = \{(Q^*, \ldots, Q^*)\}$. That is, the unique stochastically stable state has every player playing the maximum strategy. Note that all homogenous strategy profiles give payoffs of zero, so there is no welfare loss.

Consider strategic substitutes and positive externalities. If follows from Figure 8b that $u(s_i, s_j) \geq u(s_j, s_i)$ if and only if $s_i \leq s_j$. Consequently, ordering strategies $Q^*, Q^* - 1, \ldots, 0$, the reachability graph R has blocks that consist of a single element. Therefore, SCCs are the singletons $\{s_i\}$, $s_i = 0, \ldots, Q^*$, with unique sink SCC $\{0\}$. Consequently, Lemma 3 implies that, under the unperturbed process P^0, from any profile at which at least one player plays 0, the profile $(0, \ldots, 0)$ is reached in finite time with probability one. Lemma 4 implies that, if P is competitive, then $SS = \{(0, \ldots, 0)\}$. That is, the unique stochastically stable state has every player playing the minimum strategy. Note that the welfare-maximizing strategy is $Q^*/2$, so that the strategy 0 favored by our processes is inefficiently low.

4.3 | Nash Demand Game

The Nash demand game (Nash 1953) is the canonical two-player bargaining model, modeling the division of a fixed surplus through simultaneous demands. Despite its simple formulation, it captures the central tension of bargaining: high demands yield higher payoffs if accepted, but negotiations can break down if both players are too aggressive. As such, it provides a natural bridge between cooperative bargaining theory and noncooperative evolutionary analysis, and serves as a benchmark for studying efficiency, fairness, and equilibrium selection (Young 1993b; Agastya 1997; Newton 2012). In this section, we apply our model to a discrete version of the Nash demand game.

Consider $V = \{0, 1, \ldots, Q^*\}$ as in Section 4.1. Consider the payoff function

$$u(s_i, s_j) = \begin{cases} s_i, & \text{if } s_i + s_j \leq Q^*, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that, for all $v \in V \setminus \{0\}$, we have

$$u(0, v) < u(v, 0), \qquad u(v, Q^*) = 0 = u(Q^*, v),$$

therefore, from 0, there are edges in E to all $v \in \{1, \ldots, Q^*\}$, but there are no edges from $v \in 1, \ldots, Q^*$ to 0. Furthermore, there are edges from all $v \in 1, \ldots, Q^*$ to Q^* and from Q^* to all $v \in 1, \ldots, Q^*$.

Therefore, the reachability graph R has two blocks corresponding to two SCCs, $\{0\}$ and the sink SCC $V^* = \{1, \ldots, Q^*\}$. Therefore,

$$\bar{S}^* = \{(v, \ldots, v) \,:\, v \in \{1, \ldots, Q^*\}\}.$$

Consequently, Lemma 3 implies that, under the unperturbed process P^0, from any profile at which at least one player plays $v \neq 0$, some $(v, \ldots, v)$, $v \in \{1, \ldots, Q^*\}$ is reached in finite time with probability one. Lemma 4 implies that, if P is competitive, then $SS \subseteq \bar{S}^*$.

If P is homogeneous and uniform, then Corollary 2 implies that $SS = \bar{S}^*$.

If P is homogeneous and u-condition dependent, with $f(Q^*)/f(0) < n$ ensuring competitiveness, then Corollary 3 implies that SS is composed of the $(v, \ldots, v) \in \bar{S}^*$ that maximize $u(v, v)$. If Q^* is an even number, this gives $SS = \{(Q^*/2, \ldots, Q^*/2)\}$. If Q^* is an odd number, this gives $SS = \{((Q^* - 1)/2, \ldots, (Q^* - 1)/2)\}$. That is, the most efficient homogeneous strategy profile is uniquely stochastically stable.

4.4 | Two-Strategy* Games

		s_j	
		α	β
s_i	α	3	0
	β	2	2

(a) Stag hunt

		s_j	
		α	β
s_i	α	3	2
	β	4	1

(b) Hawk-dove

		s_j	
		α	β
s_i	α	3	1
	β	4	2

(c) Prisoner's dilemma

		s_j	
		α	β
s_i	α	4	2
	β	3	1

(d) Dominant efficiency

		s_j	
		α	β
s_i	α	2	0
	β	0	1

(e) Coordination

FIGURE 9 | Payoffs of symmetric two-strategy games. For example, if $s_i = \alpha$ and $s_j = \beta$, then $u(s_i, s_j) = 0$ in the Stag Hunt game.

Many important strategic situations involve binary choice (e.g., stop or go, fight or run, agree or disagree). The prevalence of such choices has helped to make two-strategy, two-player games the best studied class of games. In this section, we apply our model to the most famous two-strategy symmetric games (Prisoner's dilemma, Stag Hunt, Hawk-Dove, and Coordination) as well as to our earlier example from Figure 4.

Consider the game with strategy set $V = \{\alpha, \beta\}$ and payoffs $u(s_i, s_j)$ given by the matrices in Figure 9. The games (a)–(d) differ considerably, yet have the same weak tournament graph $G = (V, E)$,

$$E = \begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & 0 & 1 \\ \beta & 0 & 0 \end{array}, \qquad R = \begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & 1 & 1 \\ \beta & 0 & 1 \end{array}.$$

This implies that for each of these games, $V^* = \{\beta\}$. Under conventional terminology, β corresponds to the strategies *Hare* in the Stag Hunt, *Hawk* in Hawk-Dove, *Defect* in the Prisoner's dilemma, and the dominated action in the dominant efficiency game. Consequently, Lemma 3 implies that, under the unperturbed process P^0, from any profile at which at least one player plays β, the profile $(\beta, \ldots, \beta)$ is reached in finite time with probability one. Furthermore, Lemma 4 implies that, if P is competitive, then $SS = \{(\beta, \ldots, \beta)\}$.[13] Notably, in the dominant efficiency game, strategy β is strictly dominated. As discussed in the introduction, such strategies are selected against under many dynamics. However, this is not the case under PII, where it outperforms α due to "winning" in pairwise interactions.

The game (e) in Figure 9 has weak tournament graph $G = (V, E)$,

$$E = \begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & 0 & 1 \\ \beta & 1 & 0 \end{array}, \qquad R = \begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & 1 & 1 \\ \beta & 1 & 1 \end{array}.$$

Consequently, Lemma 3 implies that, under the unperturbed process P^0, $\bar{S}^* = \{(\alpha, \ldots, \alpha), (\beta, \ldots, \beta)\}$ is reached in finite time with probability one. Furthermore, Lemma 4 implies that, if P is competitive, then $SS \subseteq \bar{S}^*$.

For uniform perturbations, under the conditions of Corollary 2, we have that, for all $s \in \bar{S}^*$, $c(s) = d(s) = \delta$, where the first equality follows from homogeneity and the second from uniformity. Therefore, $OS(\bar{S}^*) = \bar{S}^*$ and Theorem 1 gives $SS = OS(\bar{S}^*) = \bar{S}^*$.

$$\begin{array}{cc|c|c|c|}
 & & & s_j & \\
 & & S & M & L \\
\hline
 & S & 7 & 5 & 5 \\
\hline
s_i & M & 8 & 6 & 4 \\
\hline
 & L & 7 & 7 & 5 \\
\hline
\end{array}$$

FIGURE 10 | Payoffs of the Irish elk game.

For u-condition-dependent perturbations, under the conditions of Corollary 3, with $f(2)/f(0) < n$ ensuring competitiveness, for $s = (v', \dots, v') \in \bar{S}^*$, for arbitrary $i \in N$,

$$c(s) = d_i(s) = f\left(\min_{v \in U_i(s)} u(s_i, v)\right) \underbrace{=}_{\substack{\text{by } s_i = v' \\ \text{and } U_i(s) = \{v'\}}} f(u(v', v')).$$

Therefore, $c((\alpha, \dots, \alpha)) = f(u(\alpha, \alpha)) = f(2)$ and $c((\beta, \dots, \beta)) = f(u(\beta, \beta)) = f(1)$. Consequently, $OS(\bar{S}^*) = \{(\alpha, \dots, \alpha)\}$, so Theorem 1 implies $SS = OS(\bar{S}^*) = \{(\alpha, \dots, \alpha)\}$.

4.5 | Sexual/Social Selection

Under our dynamics, payoffs from pairwise matches are a key factor in determining stochastic stability. Such dynamics resemble settings in which competition to be chosen as a mate or social partner plays an important role in evolutionary selection. In such settings, winning against an opponent in a pairwise match may be more important for survival than any direct fitness benefits that accrue from payoffs.

It is known that sexual/social selection may result in extreme traits, which we illustrate here in the context of our model. Recall, from Remark 1, that Bilancini et al. (2021) say that a strategy $v \in V$ is supreme if $u(v, v') > u(v', v)$ for all $v' \in V \setminus \{v\}$. Also recall that supremacy of v implies $V^* = v$. Lemma 4 then implies that, if P is competitive, then $SS = \{(v, \dots, v)\}$. That is, the unique stochastically stable state has every player playing the supreme strategy.

Example4 (Irish elk). Irish elk is an extinct species, famous for the large antlers of its males. Gould (1977) discusses how these antlers were used for ritualized combat among males. The ritualized combat prevented actual battle and established hierarchies of males that determined their status and access to females.

Let $V = \{S, M, L\}$, with strategies representing small, medium, and large antlers, respectively. The antlers were used for sexual display. A larger antler is more appealing but more costly. If both elk adopt the same strategy, each gets a benefit of 7. If they adopt differing strategies, then the larger of the two gets 9 and the smaller of the two gets 5. Assume that strategy S costs 0, M costs 1, and L costs 2. Payoffs equal benefit minus cost and are given in Figure 10.

Observe that L is supreme, so we have $V^* = \{L\}$. Lemma 4 then implies that, if P is competitive, we have $SS = \{(L, \dots, L)\}$. Consistent with this model, antlers did indeed grow large over the course of evolution. Some elk had antlers spanning up to 3.5 m. Large antlers require considerable energy to grow and make it difficult to escape from hunters in forested landscapes. Consequently, the extinction of the species is sometimes attributed to the size of its antlers.

4.6 | Contests/Conspicuous Consumption

Contests and conspicuous consumption both involve settings in which relative standing matters. Conspicuous consumption refers to the purchase of luxury goods and services to showcase one's wealth and social status (Veblen 1899), and it is defined by how an individual's spending compares with that of others. This section builds on the conspicuous consumption model of Hwang and Lee (2017). Players are consumers. Each player has income m. The strategy of player i is the amount $s_i \in V \subseteq [0, m]$ of consumption allocated to a conspicuous good. The remainder $m - s_i$ is spent on an inconspicuous good. Payoffs are

$$u(s_i, s_j) = \phi(s_i) + (m - s_i) + \varsigma \frac{f(s_j)}{f(s_i) + f(s_j)}. \tag{15}$$

The first two terms in (15) are a standard quasi-linear utility function in the two goods. Assume ϕ is positive, increasing, concave, and $\phi'(0) > 1 > \phi'(m)$, so that it is optimal to consume positive amounts of both goods.

The last term in (15) is utility from a status contest with an opposing player. The relative importance of status preference is $\varsigma > 0$. The reader may recognize f as a contest success function. Assume f is increasing, concave, $f(0) = 0$,

$$\lim_{t \to 0} \frac{f'(t)}{f(t)} = \infty, \qquad \frac{f'(m)}{f(m)} < \frac{2}{\varsigma}(1 - \phi'(m)). \tag{16}$$

Under these assumptions, we can define s^* to satisfy

$$\frac{f'(s^*)}{f(s^*)} = \frac{2}{\varsigma}(1 - \phi'(s^*)). \tag{17}$$

Finally, as the current paper deals with finite strategy sets, let

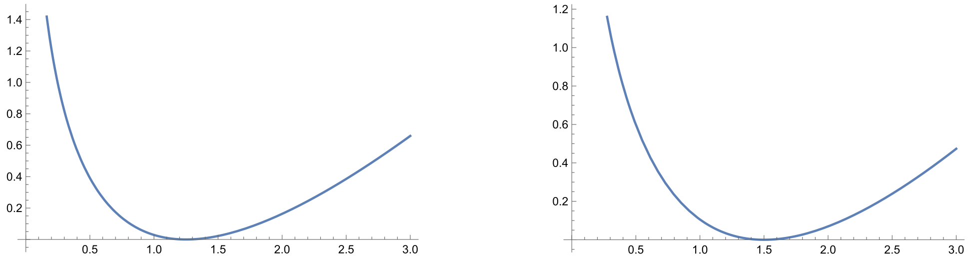
$$V = \{0, \delta, 2\delta, \dots, m - 2\delta, m - \delta, m\} \cup \{s^*\} \quad \text{for some small } \delta > 0.$$

Consider $u(s^*, t) - u(t, s^*)$, $t \in [0, m]$. It can be checked that this function is convex and uniquely minimized at $t = s^*$ (Figure 11). From $V \subseteq [0, m]$, it follows that s^* is also the minimizer on the domain V. From this, we have that $u(s^*, v') > u(v', s^*)$ for all $v' \in V \setminus \{s^*\}$, therefore $V^* = \{s^*\}$ (see Remark 1). Lemma 4 then implies that, if P is competitive, we have $SS = \{(s^*, \dots, s^*)\}$.

Notably, s^* involves more consumption of the conspicuous good than does the Nash equilibrium of this game, which solves

$$\frac{f'(s)}{f(s)} = \frac{4}{\varsigma}(1 - \phi'(s)). \tag{18}$$

Thus, in a society where consumers mimic other consumers, conspicuous consumption is higher than in a society in which



(a) $f(t) = \sqrt{t}$, $s^* = \frac{5}{4}$. **(b)** $f(t) = t$, $s^* = \frac{3}{2}$.

FIGURE 11 | Plots of $u(s^*, t) - u(t, s^*)$, $t \in [0, m]$. Parameters $\varsigma = 1$, $m = 3$, $\phi(t) = \log(t)$ and alternative specifications of $f(t)$.

TABLE 2 | Examples and applications. Stochastically stable strategies under perturbed PII, with efficient and Nash equilibrium strategies for comparison.

	Symmetric efficiency	**Symmetric Nash equ.**	***SS* under PII & Uniform**	***SS* under PII & *u*-CD**
Figure 1	α	α	β, γ	β
Cournot competition	$\frac{Q^*}{4}$	$\frac{Q^*}{3}$	$\frac{Q^*}{2}$	$\frac{Q^*}{2}$
Strategic Sub. Negative Ext.	$\frac{Q^*}{4}$	$\frac{Q^*}{3}$	$\frac{Q^*}{2}$	$\frac{Q^*}{2}$
Strategic Sub. Positive Ext.	$\frac{Q^*}{2}$	$\frac{Q^*}{3}$	0	0
Strategic Comp. Negative Ext.	$s : s_i = s_j$	Q^*	Q^*	Q^*
Strategic Comp. Positive Ext.	Q^*	Q^*	$\frac{Q^*}{2}$	$\frac{Q^*}{2}$
Nash demand game	$\frac{Q^*}{2}$	$\frac{Q^*}{2}$	$1, \ldots, Q^*$	$\frac{Q^*}{2}$
Irish elk	S	S	L	L
Status contests $f(t) = t$, $\phi(t) = \log(t)$	1	$1 + \frac{\zeta}{4}$	$1 + \frac{\zeta}{2}$	$1 + \frac{\zeta}{2}$

consumers play Nash equilibrium. Table 2 summarizes the examples and applications discussed in this paper.

5 | Discussion

5.1 | Strict Payoff Inequality for Imitation

Consider replacing the weak inequality on the right-hand side of (2) with a strict inequality, so that a player currently playing v would only switch to v' under PII if $u(v, v') < u(v', v)$. This prohibits neutral drift between tied strategies. Consequently, Lemma 1 no longer holds. To see this, consider the game in Figure 9e. This game has payoff ties, $u(\alpha, \beta) = u(\beta, \alpha)$. Consider a heterogeneous strategy profile at which some players play α and others play β. Neither strategy does strictly better than the other when they face each other, therefore there are no switches from α to β or vice versa. The profile is absorbing despite being heterogeneous.

5.2 | Asynchronous Versus Synchronous Updating

Under P^ε, only one player each period updates his strategy. Consider an alternative in which every player updates every period. This will not change our results for the following reasons. (i) As any updating player remains playing their current strategy with positive probability under PII, any transition that is possible under P^ε, $\varepsilon \in [0, 1)$, occurs with similar probability (i.e., identical cost) under simultaneous updating. Considering $\varepsilon = 0$, this implies that Lemma 1 holds. (ii) Although transitions can occur under simultaneous updating that are impossible under sequential updating, their existence does not create new transition paths between profiles in $\bar{S}$ with lower total cost than the most likely paths under P^ε. Therefore, stability analysis is unaffected.

5.3 | More Complex Payoff Comparisons

The mapping from PII to tournament graphs can be extended to payoff comparisons more complex than the comparison $u(s_i, s_j) \leq u(s_j, s_i)$ in (2). For example, we could also allow consideration of how well strategies s_i and s_j do against other strategies. The updating player might switch from s_i to s_j if

$$\rho\, u(s_i, s_j) + (1 - \rho) \sum_{v \in V} \lambda_{s_i, v}\, u(s_i, v) \leq \rho\, u(s_j, s_i) + (1 - \rho) \sum_{v \in V} \lambda_{s_j, v}\, u(s_j, v), \tag{19}$$

where $\rho \in [0, 1]$ is the weight the assessment places on the current interaction and $\lambda_{v, v'}$ is the importance assigned to v' when assessing strategy v. Importantly, the current frequency of

		Intruder	
		H	D
Owner	H	$\frac{V-C}{2}, \frac{v-C}{2}$	$V, 0$
	D	$0, v$	$\frac{V}{2}, \frac{v}{2}$

(a) The Hawk-Dove game with ownership

	HH	HD	DH	DD
HH	$\frac{V+v}{4} - \frac{C}{2}$	$\frac{V}{2} + \frac{v-C}{4}$	$\frac{V-C}{4} + \frac{v}{2}$	$\frac{V+v}{2}$
HD	$\frac{V-C}{4}$	$\frac{V}{2}$	$\frac{V+v-C}{4}$	$\frac{V}{2} + \frac{v}{4}$
DH	$\frac{v-C}{4}$	$\frac{V+v-C}{4}$	$\frac{v}{2}$	$\frac{V}{4} + \frac{v}{2}$
DD	0	$\frac{V}{4}$	$\frac{v}{4}$	$\frac{V+v}{4}$

(b) Symmetric Hawk-Dove game with ownership

FIGURE 12 | Panel (a) gives an asymmetric Hawk-Dove game, $0 < v < V < C$. Panel (b) gives a symmetric version of the game in which each letter indicates a strategy for the game in (a). The first letter indicates the strategy of the player when he is in the owner role, and the second letter indicates the strategy when he is in the intruder role. Payoffs in (b) are the expected payoffs when each of the two players is equally likely to be the owner or the intruder.

strategies does not affect the comparison. If it did, then we would have a family of tournament graphs dependent on s.

5.4 | Asymmetric Games

Our results can be extended to asymmetric games by making them symmetric. Consider the asymmetric Hawk-Dove game in Figure 12a. The asymmetry comes from two roles, an owner and an intruder, with the owner valuing the prize more than the intruder values the prize ($V > v$). Van Damme (1991) turns this game into a symmetric game by specifying meta-strategies that give strategies for both possible roles. Payoffs are then the expected payoffs from playing these contingent strategies when a player is equally likely to be allocated to each role. For the symmetric game in Figure 12, note that HH is supreme (see Remark 1), therefore $V^* = \{HH\}$.

5.5 | More Than Two-Player Games

Consider an extension of PII so that the underlying symmetric game has three players rather than two. Each period, let the updating player be matched to two other players to play the game. With positive probability, the updating player then imitates the strategy of a player who obtains the highest payoff in the game.

Consider such a game with two strategies. Let payoffs be such that if players' strategies are not all the same, then the minority player obtains the highest payoff. Using the obvious notation, with strategies $v \neq v'$,

$$u(v, v', v') \underbrace{>}_{\substack{minority\\advantage}} u(v', v, v') \underbrace{=}_{\substack{symmetric\\game}} u(v', v', v).$$

Consider a population size of $n = 3$. Starting from a heterogeneous profile, if the player with the minority strategy is selected to update, he will not change his strategy. If a player with the majority strategy is chosen to update, he will switch to the minority strategy, making it a majority. The situation is now as it was before, but with minority and majority swapped. In the absence of perturbations, such a situation can persist indefinitely. Extending this line of thought, we observe that for arbitrary $n \geq 3$, neither strategy will ever be eliminated from the population. There is no analog of Lemma 1.

6 | Summary

We investigate perturbed evolutionary dynamics based on pairwise interaction and imitation. Using a weak tournament graph representation of strategic interaction, we show that stochastically stable strategy profiles are homogenous in strategies that belong to the sink SCC of the graph. Furthermore, for generalized condition-dependent perturbations, a class that includes uniform perturbations and standard condition dependence where perturbation probabilities decrease with payoffs, we establish a one-shot stability result—stochastically stable strategy profiles coincide with homogeneous states in sink SCC strategies that are most resilient to a single deviation.

Applications are explored across various strategic settings, including Cournot competition, strategic substitutes versus complements, negative versus positive externalities, the Nash demand game, two-strategy games, sexual selection, and status-seeking contests via conspicuous consumption (see Table 2). The analysis reveals general principles such as PII's favoring of less efficient strategies in competitive environments due to spite effects. This effect contrasts with the role of condition dependence in favoring efficiency. The results provide insights into the role of perturbations in shaping long-run outcomes under boundedly rational behavioral rules.

Acknowledgments

Hwang's work was supported by the Institute of Information & Communications Technology Planning & Evaluation(IITP)-Global Data-X Leader HRD program grant funded by the Korea government(MSIT)(IITP-RS-2024-00440626), Sawa's work was supported by KAKENHI funding from the Japan Society for the Promotion of Science (Grant 22H00826,22H05086). We thank Ennio Bilancini, Leonardo Boncinelli, David Levine, Clara Sadler, and Joel Sobel for helpful interactions.

Endnotes

[1] See the survey of Newton (2018) for a discussion of behavioral rules and perturbation structures.

[2] These papers study perturbations from best response. To the best of our knowledge, there are not yet similar experimental studies for imitative dynamics.

[3] This is similar to the result of Vega-Redondo (1997), in which an imitative process drives the emergence of Walrasian equilibrium.

[4] For more on cyclic structures in perturbed dynamics, see Newton and Sandholm (2022), Levine and Modica (2016), and Cui and Zhai (2010). Under uniform perturbations, this corresponds to the *mutation-connected component* of Noldeke and Samuelson (1993).

[5] This places the paper within the body of work that links the evolution of conventions under different behavioral rules with solution concepts from noncooperative and cooperative game theory. For a demonstration of this approach, see Hwang et al. (2018) who classify bargaining solutions according to the behavioral rules that generate them. For a survey, see Newton (2018).

[6] "Rawlsian stable matching" and "Rawlsian Nash equilibrium" mean the stable matching or Nash equilibrium that gives the highest payoff to the least well-off player.

[7] Alós-Ferrer and Shi (2012) combine the efficiency effects of long memory (Bergin and Bernhardt 2009) with the spite effects of short memory (Vega-Redondo 1997) and find that spite dominates in oligopoly games, whereas efficiency dominates in Coordination games.

[8] The differing results across these groups of models illustrate the classic effects of assortative matching in promoting efficiency (Bergstrom 1995; Alger and Weibull 2013; Newton 2017). For a detailed study of evolutionary games and matching rules, see Jensen and Rigos (2018).

[9] For imitation under uniform perturbations, Fudenberg and Imhof (2006);;2008) characterize limiting distributions in the rare perturbation and large population limits in terms of the probabilities of the strategy of a single mutant taking over the population under the unperturbed process.

[10] Formally, if $P_i^0(s, s') = 0$ and $P_i^{\hat{\varepsilon}}(s, s') > 0$ for some $\hat{\varepsilon} > 0$, let $\{P_i^{\varepsilon}(s, s')\}_{\varepsilon}$ satisfy

$$P_i^{\varepsilon}(s, s') = (a + o(1))\, \varepsilon^k \quad \text{for some } a > 0,\ k > 0, \tag{4}$$

where a, k may depend on s, s', i, but not on ε; and $o(1)$ represents a term that vanishes as $\varepsilon \to 0$. Regularity is satisfied by popular behavioral rules such as uniform perturbations, condition dependence, and logit choice, the first two of which we consider in the current paper.

[11] Minimality follows from strong connectedness of sink SCC V^*. For any strict subset of V^*, say $V^{**} \subset V^*$, there exists a path from any $v \in V^{**}$ to any $v' \in V^* \setminus V^{**}$. Therefore, V^{**} is not a sink SCC.

[12] In formal terms, we have asymmetry (Peski 2010) toward V^* as long as we only consider profiles at which at least one player plays a strategy in V^*. The assumption that P is competitive ensures that this asymmetry cannot be overcome by difficulties in escaping profiles at which no player plays a strategy in V^*. To show asymmetry, consider the definition of asymmetry for an individual player (Newton 2021) but applied to composite states (Newton and Sandholm 2022) defined by equivalence classes (Naono 2022). Define equivalence classes

$$s \sim s' \Leftrightarrow \{i \in N : s_i \in V^*\} = \{i \in N : s_i' \in V^*\}.$$

Denote such equivalence classes by σ and define the process

$$\bar{P}^{\varepsilon}(\sigma, \sigma') = \frac{1}{\mu^{\varepsilon}(\sigma)} \sum_{\substack{s \in \sigma \\ s' \in \sigma'}} \mu^{\varepsilon}(s)\, P^{\varepsilon}(s, s'). \tag{11}$$

Observe an isomorphism between equivalence classes and strategy profiles in a two-strategy game. Specifically, with some abuse of notation, for all σ, arbitrary $s \in \sigma$, if $s_i \in V^*$ let $\sigma_i = A$, and if $s_i \notin V^*$ let $\sigma_i = B$. Consider $\sigma, \tilde{\sigma}$ such that every player plays A at at least one of $\sigma, \tilde{\sigma}$. For some player i, let $\sigma_i = A$, $\tilde{\sigma}_i = B$. Considering limits as $\varepsilon \to 0$, it can be shown using (11) that Definition 2 (asymmetry toward A) of Newton (2021) is satisfied as long as at least one player plays A at $\tilde{\sigma}$.

[13] For the Stag Hunt game (Figure 9a) under PII with uniform perturbations (Example 1), stochastic stability of $(\beta, \ldots, \beta)$ has previously been shown by Campigotto (2021, Proposition 2).

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Appendix A: Proofs

Proof of Lemma 1. **Step 1**.Positive probability transition from arbitrary s^t to a state in $\bar{S}$.

a. Let $U^\tau = \{v \in V : \exists\, i \in N \text{ s.t. } s_i^\tau = v\} \subseteq V$, so that $|U^\tau|$ equals the number of distinct strategies played by at least one player at time τ.

b. Set $\tau = t$.

c. **Check number of distinct strategies**.If $|U^\tau| = 1$, then $s^\tau \in \bar{S}$ and we are done.If $|U^\tau| > 1$, then select $v, v' \in U^\tau$, $v \neq v'$ such that $s_i = v$, $s_j = v'$ for some $i, j \in N$.Without loss of generality, assume that $u(v, v') \leq u(v', v)$.

d. **Update from strategy** v to v'.In period τ, some $i \in N$, $s_i = v$, is chosen, with probability $\pi(i) > 0$, to update his strategy. Expression (2) implies $P_i^0(s^\tau, (v', s_{-i}^\tau)) > 0$.Let $s_i^{\tau+1} = v'$, $s_{-i}^{\tau+1} = s_{-i}^\tau$. We have that $P^0(s^\tau, s^{\tau+1}) > 0$.

e. **Check whether strategy** v **is exhausted**.If there exists $i \in N$ with $s_i^{\tau+1} = v$, then increase τ by 1 and go to (d).If there does not exist $i \in N$ with $s_i^{\tau+1} = v$, then $|U^{\tau+1}| = |U^\tau| - 1$. Increase τ by 1 and go to (c).*Note:(1d,e) eventually terminates because the number of players playing* v *decreases by 1 with each iteration.Note:(1c) eventually terminates because* $|U^\tau|$ *decreases by 1 with each iteration.*

Step 2.Convergence to $\bar{S}$ in finite time with probability one.

- Step 1 gives a positive probability sequence of transitions from any s^t to $\bar{S}$. This path has no more than $T = |N||V|$ transitions.
- Furthermore, (2) implies that $s \in \bar{S}$ is absorbing.
- Therefore, from any $s^t \in S$, there is strictly positive probability that $s^{t+T} \in \bar{S}$. Let the minimum of these probabilities across all possible starting states $s^t \in S$ be denoted ρ.
- Therefore, for all $k \in \mathbb{N}$, the probability that $s^{t+Tk} \notin \bar{S}$ is bounded above by $(1-\rho)^k \xrightarrow{k\to\infty} 0$.

☐

Proof of Lemma 2. By definition, E includes at least one directed edge between any $v, v' \in V$, $v \neq v'$. Therefore, $G = (V, E)$ is a weak tournament graph.

Choose a subgraph G' of G that has vertex set V but only one directed edge between each $v, v' \in V$. This is a (not weak) tournament graph. Theorem 4 of Harary and Moser (1966) implies that G' contains a complete path, a path in which every vertex in V is contained in at least one edge.

As G' is a subgraph of G, we have that G also contains a complete path. Let (v^{K-1}, v^K) be the final edge of such a path. This implies that, from any $v \in V$, there exists a path to v^K. That is, v^K must be an element of any sink strongly connected component (SCC). Let

$$V^* = \{v^K\} \cup \left\{v \in V : \text{ there exists a path in } G \text{ from } v^K \text{ to } v\right\}. \tag{A.1}$$

V^* satisfies the definition of a sink SCC. Moreover, it is the only such component containing v^K and therefore the only such component. ☐

Proof of Lemma 3. Consider the proof of Lemma 1.

Let $m^\tau = |\{i \in N : s_i^\tau \in V^*\}|$ be the number of players who play a strategy in V^* at time τ. The statement of Lemma 3 tells us that $m^t \geq 1$.

Claim: m^τ is nondecreasing over time under P^0.

- Consider the strategy updates from v to v' in Step 1(c).
- For m^τ to decrease it must be that $v \in V^*$, $v' \notin V^*$ and $u(v, v') \leq u(v', v)$.
- By the definition of our edge set E in $G = (V, E)$, we then have $(v, v') \in E$.
- By definition of sink SCC this implies that v' is also in the sink SCC. That is, $v' \in V^*$, a contradiction.
- Therefore, m^τ does not decrease.

As m^τ never decreases, a state $s \in \bar{S} \setminus \bar{S}^*$ will never be reached. However, by Lemma 1 we reach $\bar{S}$ in finite time with probability one. Thus the process must reach $\bar{S} \setminus (\bar{S} \setminus \bar{S}^*) = \bar{S}^*$ in finite time with probability one. ☐

Proof of Lemma 4. **Step 1**.

In this class of models (see, e.g. Theorem 12.5.1(i) of Sandholm 2010), SS is necessarily a subset of the absorbing states of P^0. Together with Lemma 1, this implies that $SS \subseteq \bar{S}$. It remains to show that $\mu^0(\bar{S}^*) = 1$.

Step 2.

Consider an arbitrary $s \in \bar{S} \setminus \bar{S}^*$. Consider a transition to $s' = (v, s_{-i})$, $v \in V^*$. By (2), $c(s, s') > 0$. Therefore, by Definition 1, we have $c(s, s') < \alpha\, n$. From s', there exists a sequence of zero cost transitions (transitions which are possible under P^0) to $s^* = (v, \ldots, v) \in \bar{S}^*$. Therefore, the total summed cost along this path from s to s^* is strictly less than $\alpha\, n$.

Step 3.

Consider an arbitrary $s^* \in \bar{S}^*$. Consider a sequence of transitions from s^* to some $s \in \bar{S} \setminus \bar{S}^*$. By Lemma 3 and its proof, there is no zero cost transition in this sequence at which a player switches from some $v^* \in V^*$ to some $v \notin V^*$. Any such transition must have positive cost, which by Definition 1 must be at least α. To reach s, every $i \in N$ must make such a switch. Therefore, there must be at least $n = |N|$ such switches in the sequence of transitions from s^* to s. The total summed cost along this path is no less than $\alpha\, n$.

Step 4.

Using the terminology of Ellison (2000), Step 2 shows that the *coradius* of $\bar{S}^*$ is strictly less that $\alpha\, n$, and Step 3 shows that the *radius* of $\bar{S}^*$ is at least as great as $\alpha\, n$. Theorem 1 of the cited paper states that if the radius of $\bar{S}^*$ is strictly greater than the coradius, we have $\mu^0(\bar{S}^*) = 1$. ☐

Proof of Theorem 1. **Step 1**.

Consider the weighted, directed graph $(S, c(\cdot,\cdot))$ with vertex set S and the weight of the edge from s to s' given by $c(s, s')$.

A *tree rooted at* s is a subgraph of $(S, c(\cdot,\cdot))$ such that there exists a unique path from every $s' \neq s$ to s and the graph has no cycles. The *stochastic potential* of s is the minimum sum of edge weights across all trees rooted at s.

If $P = \{P^\varepsilon\}_\varepsilon$ is regular and irreducible for $\varepsilon > 0$, then $s^* \in SS$ if and only if s^* minimizes stochastic potential across all strategy profiles in $s \in S$ (see, e.g., Sandholm 2010, Theorem 12.A.2).

Step 2.

Lemma 4 shows that $SS \subseteq \bar{S}^*$. If $|\bar{S}^*| = 1$, then $OS(\bar{S}^*) = \bar{S}^* = SS$ and we are done. Henceforth, consider $|\bar{S}^*| > 1$.

Let $s^0 = (v^0, \dots, v^0) \in \bar{S}^*$ minimize stochastic potential across $s \in S$.

Take a tree $\mathcal{T}^0$ rooted at s^0 with sum of edge weights equal to the stochastic potential of s^0.

Let $s' = (v', \dots, v') \in \bar{S}^*$, $s' \neq s^0$.

Step 3.

V^* being a sink SCC, $v^0, v' \in V^*$ implies that there exists a path in $G = (V, E)$ from v^0 to v'. Denote one such path $(v^k, v^{k+1}) \in E$, $k = 0, \dots, K-1$ such that $v^K = v'$.

Construct a sequence of transitions from s^0 to s' in the following way.

For $k = 0, \dots, K-1$,

- let s^{kn+1} be such that
  - $s^{kn+1}_{i^*} = v^{k+1}$ for some $i^* \in \operatorname{argmin}_{i \in N} d_i(s^{kn})$,
  - $s^{kn+1}_j = v^k$ for all $j \neq i^*$.
- Definition 2 implies that $c(s^{kn}, s^{kn+1}) = \min_{i \in N} d_i(s^{kn}) = c(s^{kn})$.
- For $t = 2, \dots, n$, let s^{kn+t} be identical to s^{kn+t-1}, except that for some player j such that $s^{kn+t-1}_j = v^k$, we have $s^{kn+t}_j = v^{k+1}$.
- $(v^k, v^{k+1}) \in E$ and (2) imply that $c(s^{kn+t-1}, s^{kn+t}) = 0 = c(s^{kn+t-1})$.

Note that $s^{Kn} = s'$ and that every transition in this sequence is a least cost transition.

Adjust $\mathcal{T}^0$ by removing all edges leaving $s^1, \dots, s^{Kn}$ and adding edges $s^\tau \to s^{\tau+1}$ for $\tau = 0, \dots, Kn-1$. The resulting graph $\mathcal{T}^1$ is a tree rooted at $s^{Kn} = s'$.

As every edge added is a least cost transition, it follows that the sum of edge weights in $\mathcal{T}^1$ is less than or equal to the sum of edge weights in $\mathcal{T}^0$ plus $c(s^0)$ minus $c(s')$.

Consider $s^0 \notin OS(\bar{S}^*)$. The definition of $OS(\bar{S}^*)$ implies we can choose s' such that $s' \in OS(\bar{S}^*)$. By definition of $OS(\bar{S}^*)$, this implies that $c(s^0) - c(s') < 0$. Therefore, the sum of edge weights of $\mathcal{T}^1$ is strictly less than the sum of edge weights of $\mathcal{T}^0$. That is, s' has strictly lower stochastic potential than s^0, a contradiction. Therefore, $s^0 \in OS(\bar{S}^*)$.

By Step 1, SS is composed of the states that minimize stochastic potential. As s^0 could be any of these states and $s^0 \in OS(\bar{S}^*)$, we have that $SS \subseteq OS(\bar{S}^*)$.

Finally, as $s^0 \in OS(\bar{S}^*)$, if $s' \in OS(\bar{S}^*)$, then it must be that $c(s^0) - c(s') = 0$ and s' has the stochastic potential as s^0. Therefore, s' also minimizes stochastic potential and $s' \in SS$. We have $SS \supseteq OS(\bar{S}^*)$. ☐